INSTRUCTOR'S SOLUTIONS MANUAL

MARK WOODARD
Furman University

CALCULUS

AND

CALCULUS
EARLY TRANSCENDENTALS

William Briggs
University of Colorado at Denver
Lyle Cochran
Whitworth University
Bernard Gillett
University of Colorado at Boulder
with the assistance of
Eric Schulz
Walla Walla Community College

PEARSON

Boston Columbus Indianapolis New York San Francisco Upper Saddle River
Amsterdam Cape Town Dubai London Madrid Milan Munich Paris Montreal Toronto
Delhi Mexico City São Paulo Sydney Hong Kong Seoul Singapore Taipei Tokyo

Reproduced by Pearson from electronic files supplied by the author.

Publishing as Pearson, 75 Arlington Street, Boston, MA 02116.

www.pearsonhighered.com

PEARSON

Contents

Differential Equations

D1.1 Basic Ideas

D1.1.1 Second-order, because the highest-order derivative appearing in the equation is second order.

D1.1.2 Linear, because the unknown function and its derivatives appear only to the first power.

D1.1.3 The equation is second-order, so we expect two arbitrary constants in the general solution.

D1.1.4 We have $y(0) = C + 10 = 5$, so $C = -5$. The solution is $y(t) = -5e^{-3t} + 10$.

D1.1.5 Yes. Note that $y'''(t) = 0$ and $y'(t) = 2$.

D1.1.6 No. $y(0) = 6 \neq 3$.

D1.1.7 Yes, it is a solution. Note that $y'(t) = -5Ce^{-5t}$, so $y'(t) + 5y(t) = 0$.

D1.1.8 Yes, it is a solution. $y'(t) = -3Ct^{-4}$, so $ty'(t) + 3y(t) = -3Ct^{-3} + 3Ct^{-3} = 0$.

D1.1.9 Yes, it is a solution. $y'(t) = 4C_1 \cos 4t - 4C_2 \sin 4t$, so $y''(t) = -16C_1 \sin 4t - 16C_2 \cos 4t$, so $y''(t) + 16y(t) = 0$.

D1.1.10 Yes, it is a solution. $y'(x) = -C_1e^{-x} + C_2e^{x}$, so $y''(x) = C_1e^{-x} + C_2e^{x}$, so $y''(x) - y(x) = 0$.

D1.1.11 Yes, it is a solution. $y'(t) = 32e^{2t}$, so $y'(t) - 2y(t) = 32e^{2t} - (32e^{2t} - 20) = 20$. Also, $y(0) = 16 - 10 = 6$.

D1.1.12 Yes, it is a solution. $y'(t) = 48t^5$, so $ty'(t) - 6y(t) = 48t^6 - 48t^6 + 18 = 18$. Also, $y(1) = 8 - 3 = 5$.

D1.1.13 Yes, it is a solution. $y'(t) = 9 \sin 3t$, so $y''(t) = 27 \cos 3t$. Thus, $y''(t) + 9y(t) = 27 \cos 3t - 27 \cos 3t = 0$. Also, $y'(0) = 0$ and $y(0) = -3$.

D1.1.14 Yes, it is a solution. $y'(x) = \frac{1}{4}(2e^{2x} + 2e^{-2x})$ and $y''(x) = \frac{1}{4}(4e^{2x} - 4e^{-2x})$. So $y''(x) - 4y(x) = 0$. Also, $y(0) = 0$ and $y'(0) = 1$.

D1.1.15 $y(t) = \int (3 + e^{-2t})\, dt = 3t - \frac{1}{2}e^{-2t} + C$.

D1.1.16 $y(t) = \int (12t^5 - 20t^4 + 2 - 6t^{-2})\, dt = 2t^6 - 4t^5 + 2t + \frac{6}{t} + C$.

D1.1.17 $y(x) = \int (4 \tan 2x - 3 \cos x)\, dx = -2 \ln |\cos 2x| - 3 \sin x + C = 2 \ln |\sec 2x| - 3 \sin x + C$.

D1.1.18 $p(x) = \int (16x^{-9} - 5 + 14x^6)\, dx = -2x^{-8} - 5x + 2x^7 + C$.

D1.1.19 $y'(t) = \int (60t^4 - 4 + 12t^{-3})\, dt = 12t^5 - 4t - 6t^{-2} + C$. $y(t) = \int (12t^5 - 4t - 6t^{-2} + C)\, dt = 2t^6 - 2t^2 + 6t^{-1} + C_1t + C_2$.

D1.1.20 $y'(t) = \int (15e^{3t} + \sin 4t)\, dt = 5e^{3t} - \frac{1}{4} \cos 4t + C_1$. $y(t) = \int (5e^{3t} - \frac{1}{4} \cos 4t + C_1)\, dt = \frac{5}{3}e^{3t} - \frac{1}{16} \sin 4t + C_1t + C_2$.

D1.1.21 $u'(x) = \int(55x^9 + 36x^7 - 21x^5 + 10x^{-3})\,dx = 5.5x^{10} + \frac{9}{2}x^8 - \frac{7}{2}x^6 - 5x^{-2} + C_1$.
$u(x) = \int(5.5x^{10} + \frac{9}{2}x^8 - \frac{7}{2}x^6 - 5x^{-2} + C)\,dx = \frac{1}{2}x^{11} + \frac{1}{2}x^9 - \frac{1}{2}x^7 + 5x^{-1} + C_1x + C_2$.

D1.1.22 $v'(x) = \int xe^x\,dx = xe^x - e^x + C_1$. $v(x) = \int(xe^x - e^x + C_1)\,dx = xe^x - e^x - e^x + C_1x + C_2 = xe^x - 2e^x + C_1x + C_2$.

D1.1.23 $y(t) = \int(1 + e^t)\,dt = t + e^t + C$. Because $y(0) = 4 = 1 + C$, we have $C = 3$. Thus, $y(t) = t + e^t + 3$.

D1.1.24 $y(t) = \int(\sin t + \cos 2t)\,dt = -\cos t + \frac{1}{2}\sin 2t + C$. Because $y(0) = 4 = -1 + C$, we have $C = 5$. Thus, $y(t) = -\cos t + \frac{1}{2}\sin 2t + 5$.

D1.1.25 $y(x) = \int(3x^2 - 3x^{-4})\,dx = x^3 + x^{-3} + C$. Because $y(1) = 0 = 1 + 1 + C$, we have $C = -2$. So $y(x) = x^3 + x^{-3} - 2$.

D1.1.26 $y(x) = \int 4\sec^2 2x\,dx = 2\tan 2x + C$. Because $y(0) = 8 = 0 + C$, we have $C = 8$. Thus, $y(x) = 2\tan 2x + 8$.

D1.1.27 $y'(t) = \int(12t - 20t^3)\,dt = 6t^2 - 5t^4 + C_1$. Because $y'(0) = 0 = 0 + C_1$, we have $C_1 = 0$. $y(t) = \int(6t^2 - 5t^4)\,dt = 2t^3 - t^5 + C_2$. Because $y(0) = 1 = 0 - 0 + C_2$, we have $C_2 = 1$. Thus, $y(t) = 2t^3 - t^5 + 1$.

D1.1.28 $u'(x) = \int(4e^{2x} - 8e^{-2x})\,dx = 2e^{2x} + 4e^{-2x} + C_1$. Because $u'(0) = 3 = 2 + 4 + C_1$, we have $C_1 = -3$. $u(x) = \int(2e^{2x} + 4e^{-2x} - 3)\,dx = e^{2x} - 2e^{-2x} - 3x + C_2$. Because $u(0) = 1 = 1 - 2 - 0 + C_2$, we have $C_2 = 2$. Thus, $u(x) = e^{2x} - 2e^{-2x} - 3x + 2$.

D1.1.29

a. $v(t) = -9.8t + 29.4$. $s(t) = -4.9t^2 + 29.4t + 30$.

b. The object reaches its high point when $-9.8t + 29.4 = 0$, or $t = \frac{29.4}{9.8} = 3$. At that time its position is $s(3) \approx 74.1$ meters.

D1.1.30

a. $v(t) = -9.8t + 49$. $s(t) = -4.9t^2 + 49t + 60$.

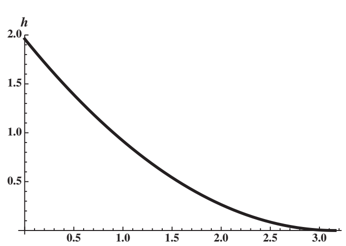
b. The object reaches its high point when $-9.8t + 49 = 0$, or $t = \frac{49}{9.8} = 5$. At that time its position is $s(5) = 182.5$ meters.

D1.1.31 We have $p(t) = (1500 - 20H)e^{.05t} + 20H$. The amount of resource is increasing when $1500 - 20H > 0$, which occurs for $H < 75$. The amount of resource is constant when $1500 - 20H = 0$, which occurs for $H = 75$. If $H = 100$, the resource is zero when $(1500 - 2000)e^{.05t} + 2000 = 0$, which occurs for $t = 20\ln 4 \approx 28$.

D1.1.32 We have $p(t) = (p_0 - 10000)e^{.05t} + 10000$. The amount of resource is decreasing when $p_0 - 10000 < 0$, or $p_0 < 10,000$. The amount of resource is constant when $p_0 = 10,000$. If $p_0 = 9000$, the resource vanishes when $-1000e^{.05t} = -10000$, or $t = 20\ln 10 \approx 46$.

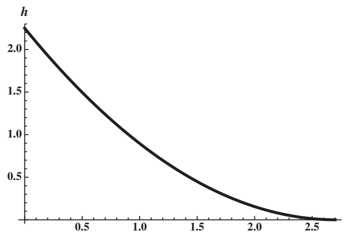
D1.1.33

The height function is given by $h(t) = \left(\sqrt{1.96} - \frac{.3\sqrt{2\cdot 9.8}}{1.5}\cdot\frac{t}{2}\right)^2 \approx (1.4 - 0.44t)^2$. The tank is empty when $h(t) = 0$, which occurs after about 3.16 seconds.



D1.1.34

The height function is given by $h(t) = \left(\sqrt{2.25} - \frac{.5\sqrt{2\cdot 9.8}}{2}\cdot\frac{t}{2}\right)^2 \approx (1.5 - .553t)^2$. The tank is empty when $h(t) = 0$, which occurs after about 2.71 seconds.



D1.1.35

a. False. That is a specific solution. The general solution is $t + C$.

b. False. It is second order, but is not linear.

c. True. First find the general solution, and then find the specific solution which satisfies the initial condition.

D1.1.36 $y(t) = \int (t\ln t + 1)\,dt = t + \int t\ln t\,dt$. Let $u = \ln t$ and $dv = t$, so that $du = \frac{dt}{t}$ and $v = t^2/2$. Then $y(t) = t + (t^2\ln t)/2 - \int t/2\,dt = t + (t^2\ln t)/2 - t^2/4 + C$.

D1.1.37 $u(x) = \int \frac{2x}{x^2+4}\,dx - \int \frac{2}{x^2+4}\,dx = \ln(x^2+4) - \tan^{-1}(x/2) + C$.

D1.1.38 Note that $\frac{4}{t^2-4} = \frac{1}{t-2} - \frac{1}{t+2}$. Thus, $v(t) = \int \frac{4}{t^2-4}\,dt = \int \left(\frac{1}{t-2} - \frac{1}{t+2}\right)\,dt = \ln\left|\frac{t-2}{t+2}\right| + C$.

D1.1.39 $y'(x) = \int \frac{x}{(1-x^2)^{3/2}}\,dx$. Let $u = 1 - x^2$, so that $du = -2x\,dx$. Substituting gives $y'(x) = \frac{-1}{2}\int u^{-3/2}\,du = u^{-1/2} + C_1 = \frac{1}{\sqrt{1-x^2}} + C_1\,dx$. $y(x) = \int \left(\frac{1}{\sqrt{1-x^2}} + C_1\right)\,dx = \sin^{-1}(x) + C_1x + C_2$.

D1.1.40 Let $u = t$ and $dv = e^t\,dt$. Then $du = dt$ and $v = e^t$. Thus, $y(t) = \int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t + C$. Because $y(0) = -1 = 0 - 1 + C$, we have $C = 0$. Thus $y(t) = te^t - e^t$.

D1.1.41 $u(x) = \int \left(\frac{1}{x^2+4^2} - 4\right)\,dx = \frac{1}{4}\tan^{-1}(x/4) - 4x + C$. Because $u(0) = 2 = 0 - 0 + C$, we have $C = 2$. Thus, $u(x) = \frac{1}{4}\tan^{-1}(x/4) - 4x + 2$.

D1.1.42 $p(x) = \int \frac{2}{x(x+1)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x+1}\right)\,dx = 2\ln\left|\frac{x}{x+1}\right| + C$. Because $p(1) = 0 = 2\ln(1/2) + C$, we have $C = -2\ln(1/2) = 2\ln 2$. Thus, $p(x) = 2\ln\left|\frac{x}{x+1}\right| + 2\ln 2$.

D1.1.43 Using the result of number 40 above, we have $y'(t) = te^t - e^t + C_1$, and because $y'(0) = 1 = 0 - 1 + C_1$, we have $C_1 = 2$. Thus $y'(t) = te^t - e^t + 2$. $y(t) = \int y'(t)\,dt = \int (te^t - e^t + 2)\,dt = te^t - e^t - e^t + 2t + C_2 = te^t - 2e^t + 2t + C_2$. Because $y(0) = 0 = 0 - 2 + 0 + C_2$, we have $C_2 = 2$. Thus, $y(t) = te^t - 2e^t + 2t + 2$.

D1.1.44 $u'(t) = Ce^{1/(4t^4)}\frac{-4}{4}t^{-5} = \frac{-u(t)}{t^5}$. Thus $u'(t) + \frac{u(t)}{t^5} = \frac{-u(t)}{t^5} + \frac{u(t)}{t^5} = 0$.

D1.1.45 $u'(t) = C_1e^t + C_2e^t + C_2te^t$, and $u''(t) = C_1e^t + C_2e^t + C_2e^t + C_2te^t = C_1e^t + 2C_2e^t + C_2te^t$. Thus, $u''(t) - 2u'(t) + u(t) = (C_1e^t + 2C_2e^t + C_2te^t) - 2(C_1e^t + C_2e^t + C_2te^t) + C_1e^t + C_2te^t = 0$.

D1.1.46 $g'(x) = -2C_1e^{-2x} + C_2e^{-2x} + -2C_2xe^{-2x}$, so $g''(x) = 4C_1e^{-2x} - 2C_2e^{-2x} + -2C_2e^{-2x} + 4C_2xe^{-2x} = 4C_1e^{-2x} - 4C_2e^{-2x} + 4C_2xe^{-2x}$. Thus, $g''(x) + 4g'(x) + 4g(x) = 4C_1e^{-2x} - 4C_2e^{-2x} + 4C_2xe^{-2x} + 4(-2C_1e^{-2x} + C_2e^{-2x} + -2C_2xe^{-2x}) + 4(C_1e^{-2x} + C_2xe^{-2x} + 2) = 8$.

D1.1.47 $u'(t) = 2C_1t + 3C_2t^2$, so $u''(t) = 2C_1 + 6C_2t$. Thus,

$$t^2u''(t) - 4tu'(t) + 6u(t) = 2C_1t^2 + 6C_2t^3 - 4(2C_1t^2 + 3C_2t^3) + 6C_1t^2 + 6C_2t^3 = 0.$$

D1.1.48 $u'(t) = 5C_1t^4 - 4C_2t^{-5} - 3t^2$, so $u''(t) = 20C_1t^3 + 20C_2t^{-6} - 6t$. Thus,

$$t^2u''(t) - 20u(t) = 20C_1t^5 + 20C_2t^{-4} - 6t^3 - 20\left(C_1t^5 + C_2t^{-4} - t^3\right) = 14t^3.$$

D1.1.49 $z'(t) = -C_1e^{-t} + 2C_2e^{2t} - 3C_3e^{-3t} - e^t$. So $z''(t) = C_1e^{-t} + 4C_2e^{2t} + 9C_3e^{-3t} - e^t$, and $z'''(t) = -C_1e^{-t} + 8C_2e^{2t} - 27C_3e^{-3t} - e^t$. Thus

$$\begin{aligned} z'''(t) + 2z''(t) - 5z'(t) - 6z(t) &= -C_1e^{-t} + 8C_2e^{2t} - 27C_3e^{-3t} - e^t \\ &+ 2C_1e^{-t} + 8C_2e^{2t} + 18C_3e^{-3t} - 2e^t \\ &+ 5C_1e^{-t} - 10C_2e^{2t} + 15C_3e^{-3t} + 5e^t \\ &- 6C_1e^{-t} - 6C_2e^{2t} - 6C_3e^{-3t} + 6e^t \\ &= 8e^t \end{aligned}$$

D1.1.50

a. $y'(t) = C_1e^t - C_2e^{-t}$, so $y''(t) = C_1e^t + C_2e^{-t}$. Thus, $y''(t) - y(t) = 0$.

b. $y'(t) = 2C_1e^{2t} - 2C_2e^{-2t}$, so $y''(t) = 4C_2e^{2t} + 4C_2e^{-2t}$. Thus, $y''(t) - 4y(t) = 0$.

c. It appears that a general solution should be $C_1e^{kt} + C_2e^{-kt}$. Then $y'(t) = kC_1e^{kt} - kC_2e^{-kt}$, and $y''(t) = k^2C_1e^{kt} + k^2C_2e^{-kt}$. Thus, $y''(t) - k^2y(t) = 0$.

d. If $y(t) = C_1\cosh kt + C_2\sinh kt$, then $y'(t) = kC_1\sinh kt + kC_2\cosh kt$ and $y''(t) = k^2C_1\cosh kt + k^2C_2\sinh kt$. Thus $y''(t) - k^2y(t) = 0$.

D1.1.51

a. $y'(t) = C_1\cos t - C_2\sin t$, so $y''(t) = -C_1\sin t - C_2\cos t$. Thus, $y''(t) + y(t) = 0$.

b. $y'(t) = 2C_2\cos 2t - 2C_2\sin 2t$, so $y''(t) = -4C_2\sin 2t - 4C_2\cos 2t$. Thus, $y''(t) + 4y(t) = 0$.

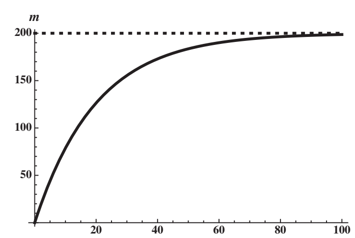
c. A general solution appears to be $y(t) = C_1\sin kt + C_2\cos kt$. Then $y'(t) = kC_1\cos kt - kC_2\sin kt$, so $y''(t) = -k^2C_1\sin kt - k^2C_2\cos kt$. And then $y''(t) + k^2y(t) = 0$.

D1.1.52

a. Let $m(t) = \frac{I}{k}(1 - e^{-kt})$. Note that $m(0) = 0$. Then $m'(t) = \frac{I}{k}(ke^{-kt})$. Therefore,

$$m'(t) + km(t) = \frac{I}{k}(ke^{-kt}) + \frac{kI}{k}(1 - e^{-kt}) = Ie^{-kt} + I - Ie^{-kt} = I.$$

b. We have $m(t) = 200(1 - e^{-.05t})$.



c. It appears that $\lim_{t\to\infty} m(t) = 200$.

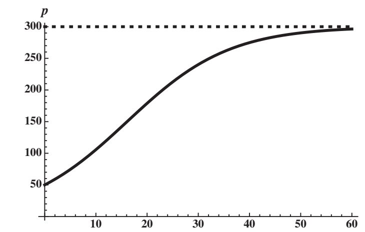
D1.1.53

a. Let $p(t) = \frac{K}{1+Ce^{-rt}}$. Note that $1 - \frac{P}{K} = 1 - \frac{1}{1+Ce^{-rt}} = \frac{Ce^{-rt}}{1+Ce^{-rt}}$. We have

$$p'(t) = \frac{KCre^{-rt}}{(1 + Ce^{-rt})^2} = r \cdot \frac{K}{1 + Ce^{-rt}} \cdot \frac{Ce^{-rt}}{1 + Ce^{-rt}} = rp\left(1 - \frac{p}{K}\right).$$

b. If $p(0) = 50 = \frac{K}{1+C}$, then $50 + 50C = K$, so $C = \frac{K-50}{50}$.

c. We have $p(t) = \frac{300}{1+5e^{-.1t}}$.



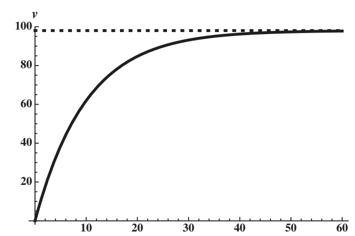
d. $\lim_{t\to\infty} \frac{300}{1+5e^{-.1t}} = \frac{300}{1+0} = 300$, which is consistent with the graph from part c.

D1.1.54

a. Let $v(t) = \frac{g}{b}(1 - e^{-bt})$. Then $v(0) = 0$, and

$$v'(t) = \frac{g}{b} \cdot be^{-bt} = ge^{-bt} = g - b \cdot \frac{g}{b}(1 - e^{-bt}) = g - bv.$$

b. With $b = 0.1$, we have $v(t) = 98(1 - e^{-.1t})$.



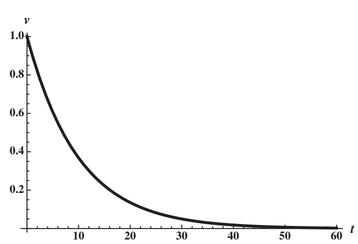
c. $\lim_{t\to\infty} v(t) = 98$.

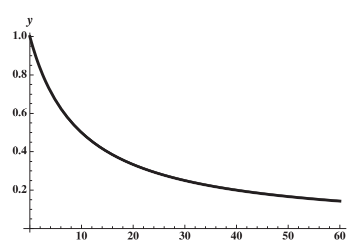
D1.1.55

a. If $y(t) = y_0 e^{-kt}$, then $y(0) = y_0$, and $y'(t) = -ky_0e^{-kt}$, so $y'(t) = -ky(t)$.

b. Let $y(t) = \frac{y_0}{y_0kt+1}$. Then $y(0) = y_0$, and $y'(t) = \frac{-y_0^2k}{(y_0kt+1)^2} = -k(y(t))^2$.

c. The first order reaction decays more quickly.



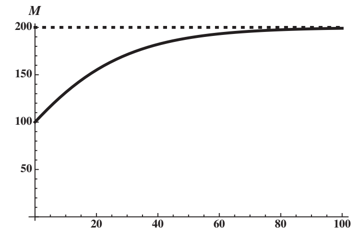


D1.1.56

a. Let $M(t) = K\left(\frac{M_0}{K}\right)^{e^{-rt}}$. Note that $\ln(M(t)/K) = e^{-rt}\ln(M_0/K)$.

$M'(t) = K\left(\frac{M_0}{K}\right)^{e^{-rt}} \ln(M_0/K)\,(-re^{-rt}) = -rM(t)\ln(M(t)/K)$. Also, $M(0) = K(M_0/K) = M_0$.

b. Using $K = 200$, $M_0 = 100$, and $r = .05$, we have $M(t) = K\left(\frac{M_0}{K}\right)^{e^{-rt}} = 200(1/2)^{e^{-.05t}}$.



c. $\lim_{t\to\infty} M(t) = 200 = K$.

D1.2 Direction Fields and Euler's Method

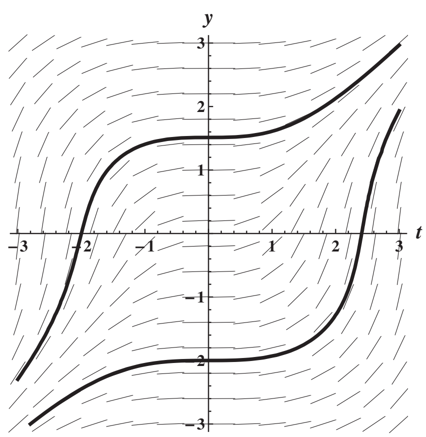
D1.2.1 Choose a regular grid of points in the ty-plane, and for each point P, make a small line segment with slope $f(t, y)$.

D1.2.2 It will have slope $3^2 - 3(1)^2 = 6$.

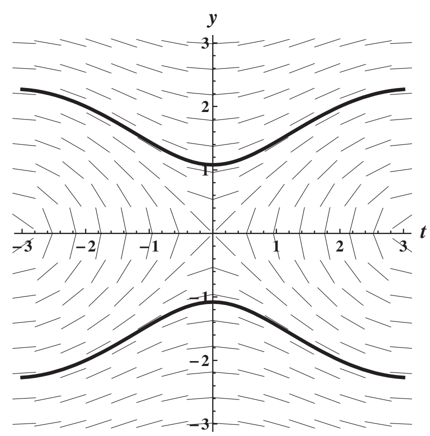
D1.2.3 $u_0 = y(3) = 1$. $u_1 = u_0 + f(3, 1)(.1) = 1 + .6 = 1.6$.

D1.2.4 Because the differential equation is giving the slope at a given point, we can approximate the solution to the differential equation by starting at the initial point, and using the slope to guide where the next iteration should be. In essence, we are numerically "following the direction field" to estimate the solution to the differential equation.

D1.2.5



D1.2.6



D1.2.7

a. This matches with D.

b. This matches with B.

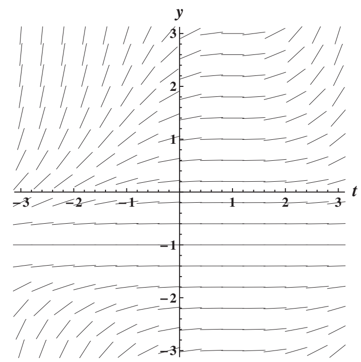
c. This matches with A.

d. This matches with C.

D1.2.8 Note that the slopes are zero when $y = -1$ and when $t = 1$. Also, they are positive when both $y > -1$ and $t > 1$. The only differential equation with this property is choice a.

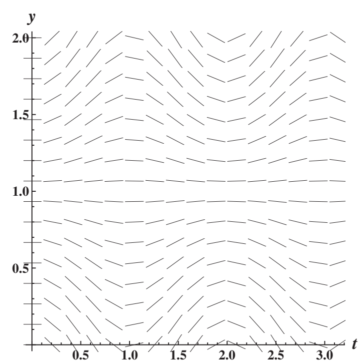
D1.2.9

An initial condition of $y(0) = -1$ leads to a constant solution. For any other initial condition, the solutions are increasing over time.



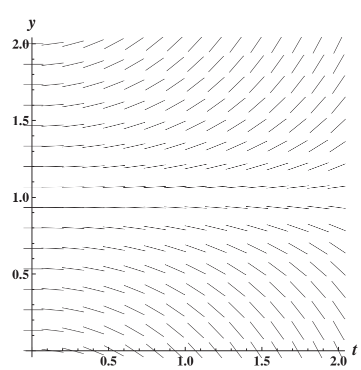
D1.2.10

An initial condition of $y(0) = 1$ leads to a constant solution. For any other initial condition, the solutions oscillate between increasing and decreasing over intervals of time length one.

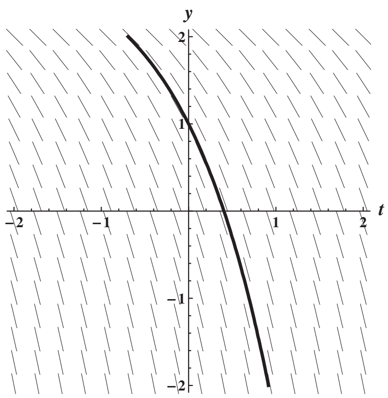


D1.2.11

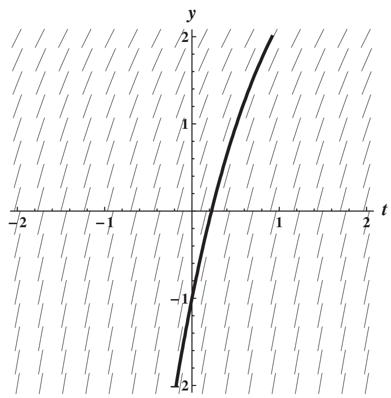
An initial condition of $y(0) = 1$ leads to a constant solution. Initial conditions $y(0) = A$ lead to solutions that are increasing over time if $A > 1$ and solutions that are decreasing over time if $A < 1$.



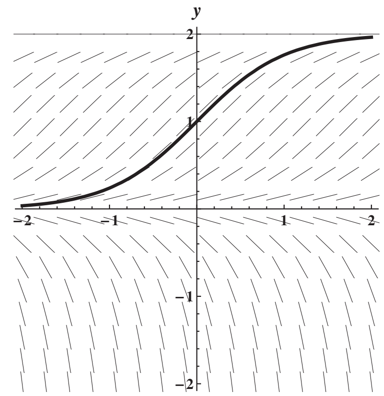
D1.2.12



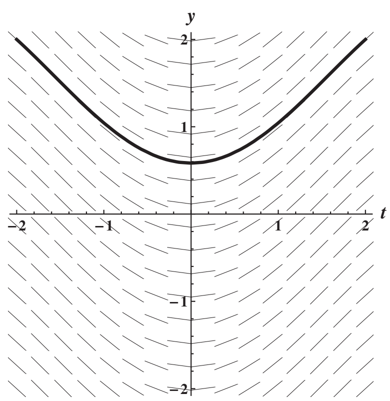
D1.2.13



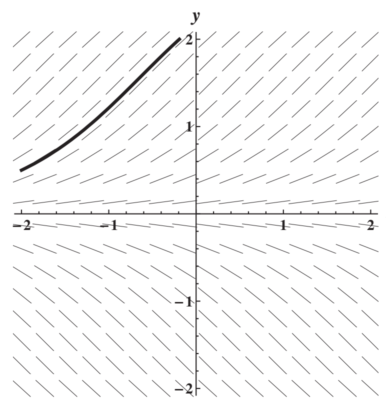
D1.2.14



D1.2.15



D1.2.16

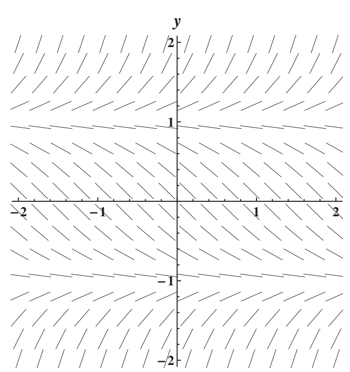


D1.2.17

a. The solutions $y = 1$ and $y = -1$ are constant.

b. Solutions are increasing when both $y > 1$ and $y > -1$, (so for $y > 1$) and when both $y < 1$ and $y < -1$ (so for $y < -1$). In other words, for $|y| > 1$. Solutions are decreasing for $|y| < 1$.

c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $|A| > 1$ and decreasing over time if $|A| < 1$.

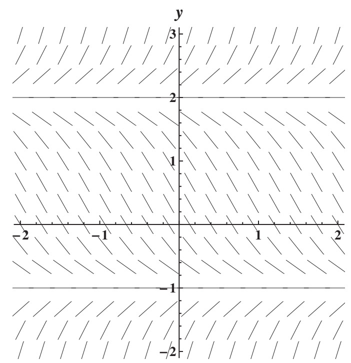
d. 

D1.2.18

a. The solutions $y = 2$ and $y = -1$ are constant.

b. Solutions are increasing when both $y > 2$ and $y > -1$, (so for $y > 2$) or when both $y < 2$ and $y < -1$ (so for $y < -1$). Solutions are decreasing for $-1 < y < 2$.

c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $A > 2$ or $A < -1$, and decreasing over time if $-1 < A < 2$.

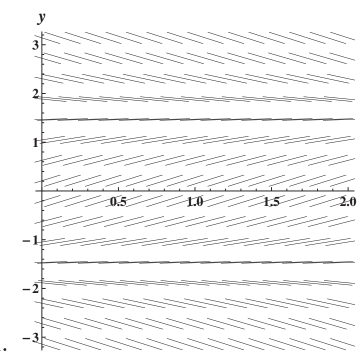
d. 

D1.2.19

a. The solutions $y = \pi/2$ and $y = -\pi/2$ are constant.

b. Solutions are increasing when $|y| < \pi/2$ and decreasing when $\pi/2 < |y| < \pi$.

c. Initial condition $y(0) = a$ leads to solutions that are increasing over time if $|A| < \pi/2$ and decreasing over time if $|A| > \pi/2$.

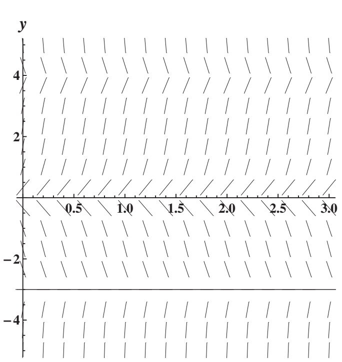
d. 

D1.2.20

a. The solutions $y = 0$, $y = -3$, and $y = 4$ are constant.

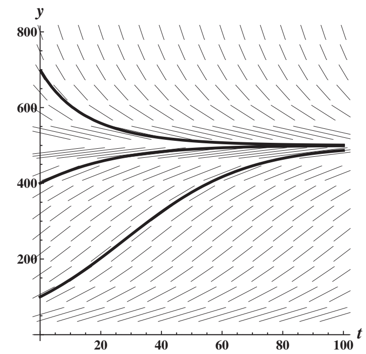
b. Solutions are increasing when $y < -3$ and when $0 < y < 4$. Solutions are decreasing when $-3 < y < 0$ and when $y > 4$.

c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $A < -3$ or $0 < A < 4$, and decreasing over time if $-3 < A < 0$ or $A > 4$.

d. 

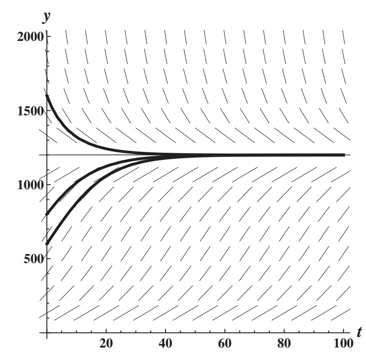
D1.2.21

Equilibrium solutions are $P(t) = 0$ and $P(t) = 500$.



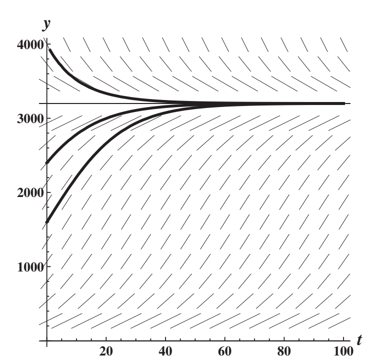
D1.2.22

Equilibrium solutions are $P(t) = 0$ and $P(t) = 1200$.



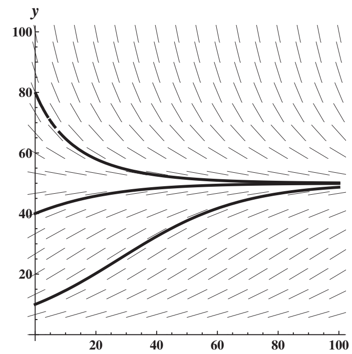
D1.2.23

Equilibrium solutions are $P(t) = 0$ and $P(t) = 3200$.



D1.2.24

Equilibrium solutions are $P(t) = 0$ and $P(t) = 50$.



D1.2.25 $u_0 = 2$. $u_1 = 2 + f(0, 2)(.5) = 2 + (4)(.5) = 4$. $u_2 = 4 + f(.5, 4)(.5) = 4 + (8)(.5) = 8$. So $y(.5) \approx 4$ and $y(1) \approx 8$.

D1.2.26 $u_0 = -1$. $u_1 = -1 + f(0, -1)(.2) = -1 + 1(.2) = -.8$. $u_2 = -.8 + f(.2, -.8)(.2) = -.8 + (.8)(.2) = -.64$. So $y(.2) \approx -.8$ and $y(.4) \approx -.64$.

D1.2.27 $u_0 = 1$. $u_1 = 1 + f(0, 1)(.1) = 1 + .1 = 1.1$. $u_2 = 1.1 + f(.1, 1.1)(.1) = 1.1 + (.9)(.1) = 1.19$. So $y(.1) \approx 1.1$ and $y(.2) \approx 1.19$.

D1.2.28 $u_0 = 4$. $u_1 = 4 + f(0, 4)(.5) = 4 + (4)(.5) = 6$. $u_2 = 6 + f(.5, 6)(.5) = 6 + (6.5)(.5) = 9.25$. So $y(.5) \approx 6$ and $y(1) \approx 9.25$.

D1.2.29

a.

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
.2	.8	.64
.1	.81	.65610
.05	.81451	.66342
.025	.81665	.66692

b. $e^{-.2} \approx 0.818731$ and $e^{-.4} \approx .67032$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
.2	.018371	.03032
.1	.00873	.01422
.05	.00422	.00690
.025	.00208	.00340

c. The time step $\Delta t = .025$ has the smallest errors. A smaller time step generally produces more accurate results.

d. Halving the time steps results in approximately halving the error.

D1.2.30

a.

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
.2	2.2	2.42
.1	2.205	2.43101
.05	2.20763	2.43681
.025	2.20897	2.43987

b. $2e^{.1} \approx 2.21034$ and $2e^{.2} \approx 2.44281$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
.2	0.0103418	0.0228055
.1	0.00534184	0.011793
.05	0.00271605	0.00599972
.025	0.00136963	0.00302642

c. The time step $\Delta t = .025$ has the smallest errors. A smaller time step generally produces more accurate results.

d. Halving the time steps results in approximately halving the error.

D1.2.31

a.

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
.2	3.2	3.36
.1	3.19	3.3439
.05	3.18549	3.33658
.025	3.18335	3.33308

b. $4 - e^{-.2} \approx 3.18127$ and $4 - e^{-.4} \approx 3.32968$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
.2	0.0187308	0.03032
.1	0.00873075	0.01422
.05	0.0042245	0.00689961
.025	0.0020789	0.00339988

c. The time step $\Delta t = .025$ has the smallest errors. A smaller time step generally produces more accurate results.

d. Halving the time steps results in approximately halving the error.

D1.2.32

a.

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
.2	.2	.48
.1	.22	.52
.05	.23	.54
.025	.235	.55

b. $.2^2 + .2 = .24$ and $.4^2 + .4 = .56$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
.2	0.04	0.08
.1	0.02	0.04
.05	0.01	0.02
.025	0.005	0.01

c. The time step $\Delta t = .025$ has the smallest errors. A smaller time step generally produces more accurate results.

d. Halving the time steps results in approximately halving the error.

D1.2.33

a. The computations yield:

t_k	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
u_k	1	0.6	0.36	0.216	0.1296	0.07776	0.046656	0.0279936	0.0167962	0.0100777	0.00604662

So $y(2) \approx 0.00604662$.

b. $y(2) = e^{-4} \approx 0.0183156$, so the error is about $0.0183156 - 0.00604662 = 0.012269$.

c. The computations yield:

t_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
u_k	1	0.8	0.64	0.512	0.4096	0.32768	0.262144	0.209715	0.167772	0.134218	0.107374

t_k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
u_k	0.0859	0.0687	0.0550	0.0440	0.0352	0.02815	0.02252	0.01801	0.01441	0.01153

So $y(2) \approx 0.01153$ The error is about $0.0183156 - 0.01153 = 0.0067856$.

d. The error with twice as many steps is about half the other error.

D1.2.34

a. The computations yield:

t_k	0	0.2	0.4	0.6	0.8	1	1.2	1.4
u_k	-1	0.6	1.56	2.136	2.4816	2.68896	2.81338	2.88803

t_k	1.6	1.8	2	2.2	2.4	2.6	2.8	3
u_k	2.93282	2.95969	2.97581	2.98549	2.99129	2.99478	2.99687	2.99812

So $y(3) \approx 2.99812$.

b. $y(3) = (3 - 4e^{-6)} \approx 2.99008$, so the error is about $2.99812 - 2.99008 = 0.00804$.

c. The computations yield:

t_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
u_k	-1	-0.2	0.44	0.952	1.3616	1.68928	1.95142	2.16114	2.32891	2.46313	2.5705

t_k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
u_k	2.6564	2.72512	2.7801	2.82408	2.85926	2.88741	2.90993	2.92794	2.94235	2.95388

t_k	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
u_k	2.96311	2.97049	2.97639	2.98111	2.98489	2.98791	2.99033	2.99226	2.99381	2.99505

So $y(3) \approx 2.99505$ The error is about $2.99812 - 2.99505 = 0.00307$.

d. The error with twice as many steps is less than half the other error.

D1.2.35

a. After many calculations, we arrive at $y(4) \approx 3.05765$.

b. $y(4) = 3 + 5e^{-4} \approx 3.09158$, so the error is about $3.09158 - 3.05765 = .03393$.

c. After many calculations, we arrive at $y(4) = 3.0739$. The error is about $3.09158 - 3.0739 = 0.01768$.

d. The error with twice as many steps is about half the other error.

D1.2.36

a. After many calculations, we arrive at $y(2) \approx 4.45125$.

b. $y(4) = \sqrt{20} \approx 4.47214$, so the error is about $4.47214 - 4.45125 = .02089$.

c. After many calculations, we arrive at $y(2) = 4.46173$. The error is about $4.47214 - 4.46173 = 0.01041$.

d. The error with twice as many steps is about half the other error.

D1.2.37

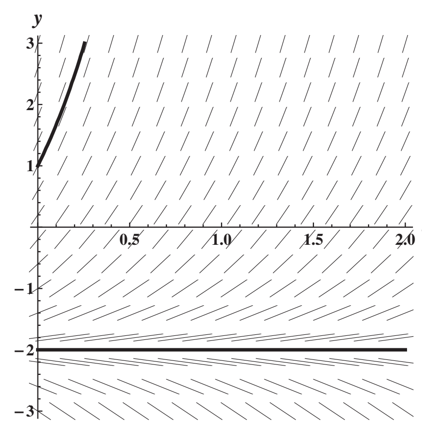
a. True.

b. False. It allows you to compute approximations.

D1.2.38

a. $y = -2$ is an equilibrium solution, because $2(-2) + 4 = 0$.

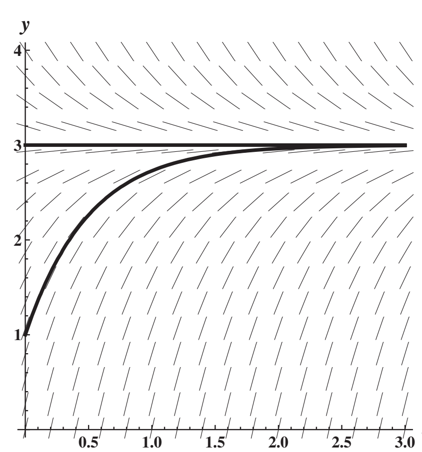
b and c.



D1.2.39

a. $y = 3$ is an equilibrium solution, because $6 - 2(3) = 0$.

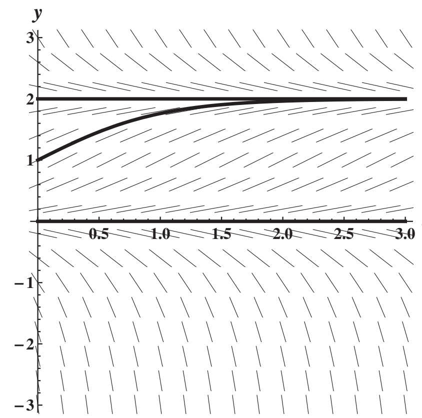
b and c.



D1.2.40

a. Solve $y(2-y) = 0$ to get equilibrium solutions $y = 0$ and $y = 2$.

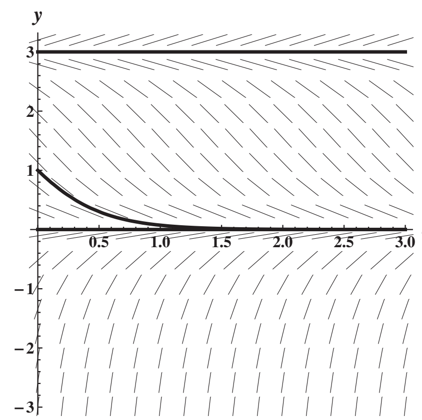
b and c.



D1.2.41

a. Solve $y(y-3) = 0$ to get equilibrium solutions $y = 0$ and $y = 3$.

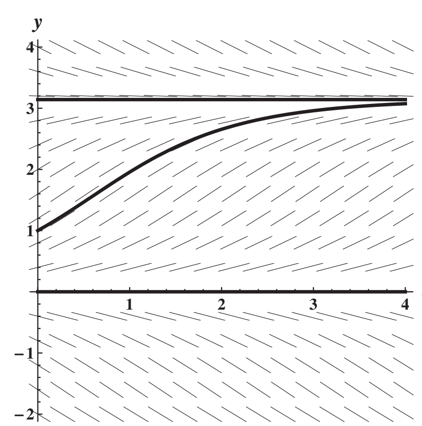
b and c.



D1.2.42

a. Solve $\sin y = 0$ to get equilibrium solutions $y = k\pi$, where k is any integer.

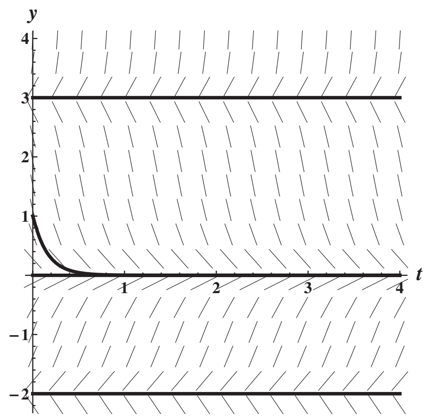
b and c.



D1.2.43

a. The equilibrium solutions are $y = 0$, $y = -2$ and $y = 3$.

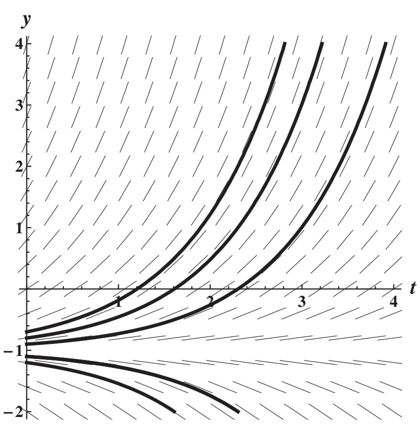
b and c.



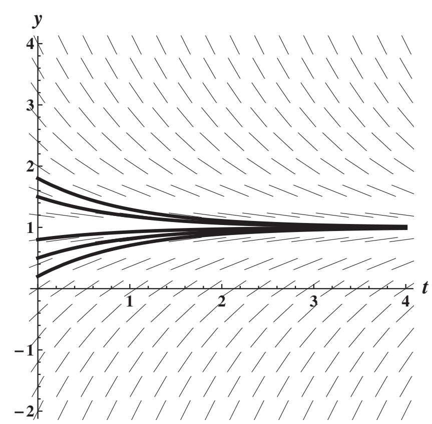
D1.2.44

a. Solving $y' = 0$ gives the equilibrium solution $y = -b/a$, which is a horizontal line.

b.



c.



Note that the general solutions $y = (A + \frac{b}{a})e^{at} - \frac{b}{a}$ increases without bound if $a > 0$ and $A > \frac{-b}{a}$, and decreases without bound if $a > 0$ but $A < \frac{-b}{a}$. But if $a < 0$, the general solutions have limit $\frac{-b}{a}$, but must increase to it if $A < \frac{-b}{a}$ and decrease to it if $A > \frac{-b}{a}$.

D1.2.45

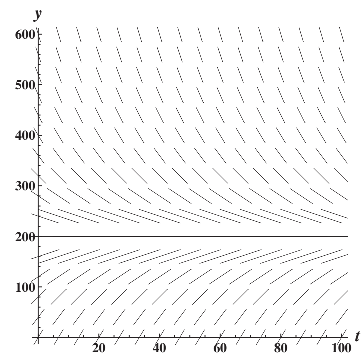
a. $\Delta t = \frac{b-a}{N}$.

b. Recall that $u_0 = A$ and $t_0 = a$. So $u_1 = A + f(a, A)\left(\frac{b-a}{N}\right)$.

c. $u_{k+1} = u_k + f(t_k, u_k)\left(\frac{b-a}{N}\right)$, where $t_k = a + k\left(\frac{b-a}{N}\right)$ for $k = 0, 1, 2, \ldots, N-1$.

D1.2.46

a.

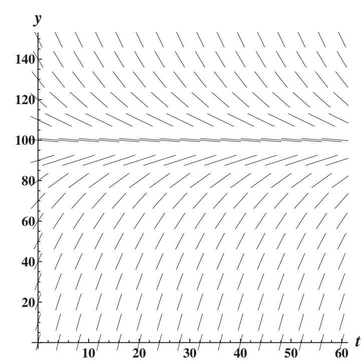


b. The equilibrium solution is $m(t) = 200$.

c. The solutions are increasing for $A < 200$ and decreasing for $A > 200$.

D1.2.47

a.

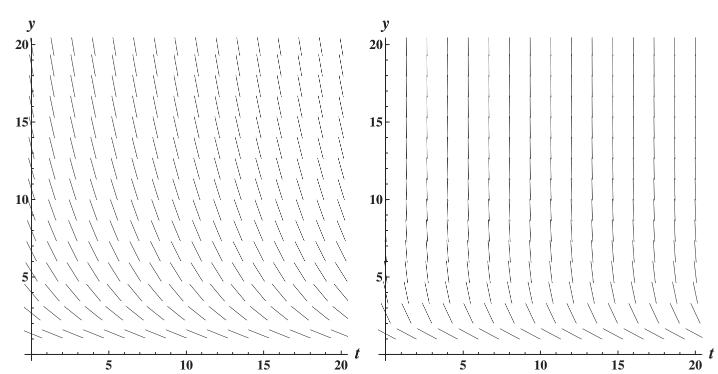


b. The solutions are increasing for $A < 98$ and decreasing for $A > 98$.

c. The equilibrium solution is $v(t) = 98$.

D1.2.48

a. In both cases, the equilibrium is 0.



b. The second order approaches more quickly.

D1.2.49

a. We have $u_0 = y(0) = 1$, and $u_{k+1} = u_k + f(t_k, u_k)h = u_k + au_kh = u_k(1 + ah)$ for $k = 0, 1, 2 \ldots$.

b. Suppose $u_k = (1 + ah)^k$. Then $u_{k+1} = u_k(1 + ah) = (1 + ah)^k(1 + ah) = (1 + ah)^{k+1}$, $k = 0, 1, 2, \ldots$.

c. $\lim_{h\to 0} u_k = \lim_{h\to 0}(1 + ah)^k = \lim_{h\to 0}(1 + ah)^{t_k/h} = \lim_{h\to 0} \left((1 + ah)^{1/h}\right)^{t_k} = (e^a)^{t_k} = e^{at_k} = y(t_k)$.

D1.2.50

a. We have $u_0 = y(0) = 1$, and $u_{k+1} = u_k + f(t_k, u_k)h = u_k - au_kh = u_k(1 - ah)$ for $k = 0, 1, 2 \ldots$.

b. Suppose $u_k = (1 - ah)^k$. Then $u_{k+1} = u_k(1 - ah) = (1 - ah)^k(1 - ah) = (1 - ah)^{k+1}$, $k = 0, 1, 2, \ldots$.

c. The function r^z increases as z increases exactly when $|r| > 1$ and decreases exactly when $|r| < 1$. So $u_k = (1-ah)^k$ will increase when k increases exactly when $|1-ah| > 1$ and will decrease as k increases exactly when $|1 - ah| < 1$.

d. Suppose $|1 - ah| < 1$. Then $-1 < 1 - ah < 1$. Subtracting one from everything gives $-2 < -ah < 0$. Dividing through by $-a$ (which is a negative number) gives $0 < h < 2/a$, as desired.

D1.3 Separable Differential Equations

D1.3.1 A separable first-order differential equation is one that can be written in the form $g(y)y'(t) = h(t)$, where the factor $g(y)$ is a function of y and $h(t)$ is a function of t.

D1.3.2 Yes, this equation is separable because it can be written in the form $y^2y'(t) = t^{-2}(t + 4)$.

D1.3.3 No, this equation cannot be written in the required form.

D1.3.4 Integrate both sides with respect to t and convert the integral on the left side to an integral with respect to y.

D1.3.5 We have $y'(t) = t^3$, so $\int \frac{dy}{dt}\,dt = \int t^3\,dt$, so $y = t^4/4 + C$.

D1.3.6 $y'(t) = 5e^{-4t}$, so $\int y'(t)\,dt = \int 5e^{-4t}\,dt$, so $y = \frac{-5}{4}e^{-4t} + C$.

D1.3.7 $y\frac{dy}{dt} = 3t^2$, so $\int y\,dy = \int 3t^2\,dt$. Thus, $\frac{y^2}{2} = t^3 + C$, and thus $y = \pm\sqrt{2t^3 + C}$.

D1.3.8 We have $\frac{1}{y}\frac{dy}{dx} = x^2 + 1$, so $\int \frac{1}{y}\frac{dy}{dx}\,dx = \int(x^2 + 1)\,dx$. Thus, $\ln y = x^3/3 + x + C$, and $y = Ae^{(x^3/3)+x}$.

D1.3.9 We have $\int e^{-y/2}\,dy = \int \sin t\,dt$, and so $-2e^{-y/2} = -\cos t + C$. Thus, $y = -2\ln\left(\frac{1}{2}\cos t + C\right)$.

D1.3.10 We have $\int w^{-1/2}\,dw = \int \frac{3x+1}{x^2}\,dx = \int\left(\frac{3}{x} + \frac{1}{x^2}\right)dx$, so $2w^{1/2} = 3\ln|x| - \frac{1}{x} + C$, and thus $w = \left(\frac{3}{2}\ln|x| - \frac{1}{2x} + C\right)^2$.

D1.3.11 $\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x^2}$, so $\int \frac{1}{y^2}\frac{dy}{dx}\,dx = \int \frac{1}{x^2}\,dx$, so $\frac{-1}{y} = \frac{-1}{x} + C = \frac{Cx-1}{x}$. Thus, $y = \frac{x}{1-Cx}$. If we replace the arbitrary constant C by its opposite, this can be written as $y = \frac{x}{1+Cx}$.

D1.3.12 $\frac{y}{y^2+4}\,y'(t) = \frac{t}{(t^2+1)^3}$. Therefore, $\int \frac{y}{y^2+4}\,dy = \int \frac{t}{(t^2+1)^3}\,dt$. We have $\frac{1}{2}\ln(y^2 + 4) = \frac{-1}{4}\frac{1}{(t^2+1)^2} + C$, so $\ln(y^2 + 4) = C_1 - \frac{1}{2(t^2+1)^2}$. This can be written as $y^2 + 4 = Ae^{-(t^2+1)^{-2}/2}$, so $|y| = \sqrt{Ae^{-(t^2+1)^{-2}/2} - 4}$.

D1.3.13 $\frac{-2}{y^3}\frac{dy}{dt} = \sin t$, so $\int \frac{-2}{y^3}\frac{dy}{dt}\,dt = \int \sin t\,dt$. Thus, $\frac{1}{y^2} = -\cos t + C$. Solving for y gives $y = \pm\frac{1}{\sqrt{C-\cos t}}$.

D1.3.14 $\frac{1}{y^2+4}y'(t) = e^{-t/2}$, so $\int \frac{1}{y^2+4}\,dy = \int e^{-t/2}\,dt$, so $\frac{1}{2}\tan^{-1}(y/2) = -2e^{-t/2} + C$. Thus, $\tan^{-1}(y/2) = C_1 - 4e^{-t/2}$, and $y = 2\tan(C_1 - 4e^{-t/2})$.

D1.3.15 $e^u u'(x) = e^{2x}$, so $\int e^u\,du = \int e^{2x}\,dx$, and $e^u = \frac{1}{2}e^{2x} + C$. Thus, $u = \ln\left(\frac{e^{2x}}{2} + C\right)$.

D1.3.16 $\frac{1}{u^2-4}u'(x) = \frac{1}{x}$, so $\int \frac{1}{u^2-4}\,du = \int \frac{1}{x}\,dx$, and thus $\frac{1}{4}\ln\left|\frac{u-2}{u+2}\right| = \ln|x| + C$, so $\ln\left|\frac{u-2}{u+2}\right| = 4\ln|x| + C_1$. Therefore, $\left|\frac{u-2}{u+2}\right| = Ax^4$, so $1 - \frac{4}{u+2} = \pm Ax^4$, and $\frac{4}{u-2} = 1 \pm Ax^4$, so $u - 2 = \frac{4}{1\pm Ax^4}$, and $u = 2 + \frac{4}{1\pm Ax^4}$.

D1.3.17 This is separable, and can be written as $y'(t) = \frac{1}{t}$. Thus, $\int y'(t)\,dt = \int \frac{dt}{t} = \ln t + C$, so $y(t) = \ln t + C$. Because $y(1) = 2 = 0 + C$, we have $C = 2$. Thus, $y(t) = \ln t + 2$.

D1.3.18 This is separable, and can be written as $y'(t) = \cos t$. Thus $y(t) = \sin t + C$, and because $y(0) = 1 = 0 + C$, we have $C = 1$. Thus, $y(t) = \sin t + 1$.

D1.3.19 This is separable, and is already written in the desired form. We have $\int 2y\,dy = \int 3t^2\,dt$, so $y^2 = t^3 + C$. Because $y(0) = 9$, we have $81 = C$, so $y = \sqrt{t^3 + 81}$.

D1.3.20 This equation is not separable.

D1.3.21 This equation is not separable.

D1.3.22 This equation is separable. We have $\int \frac{dy}{y} = \int (4t^3 + 1)\,dt$, and thus $\ln|y| = t^4 + t + C$. Therefore, $y = \pm e^{(t^4+t+C)} = Ae^{t^4+t}$. Substituting $y(0) = 4$ gives $A = 4$, so the solution to this initial value problem is $y = 4e^{t^4+t}$.

D1.3.23 This equation is separable. We have $\int 2y\,dy = \int e^t\,dt$, so $y^2 = e^t + C$, and thus $y = \pm\sqrt{e^t + C}$. Substituting $y(\ln 2) = 1$ gives $1 = 2 + C$ so $C = -1$, and the solution to this initial value problem is $y = \sqrt{e^t - 1}$.

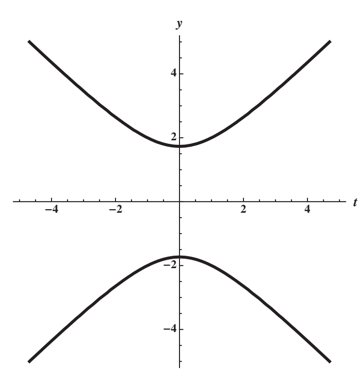
D1.3.24 This equation is separable. We have $\int y^{-3}\,dy = \int \cos x\,dx$, so $-\frac{y^{-2}}{2} = \sin x + C$. Therefore, $y = \pm(-2\sin x + C)^{-1/2}$. Substituting $y(0) = 3$ gives $C = 1/9$, so the solution to this initial value problem is $y = \left(-2\sin x + \frac{1}{9}\right)^{-1/2}$.

D1.3.25 This equation is separable. We have $\int e^y\,dy = \int e^x\,dx$, and thus $e^y = e^x + C$. Therefore, $y = \ln(e^x + C)$. Substituting $y(0) = \ln 3$ gives $\ln 3 = \ln(1 + C)$, so $C = 2$ and the solution to this initial value problem is $y = \ln(e^x + 2)$.

D1.3.26 This equation is separable. We have $\int \sec^2 y\,dy = \int dt$, so $\tan y = t + C$. Because $y(1) = \pi/4$ we have $1 = 1 + C$, so $C = 0$. Thus, $y = \tan^{-1}(t)$.

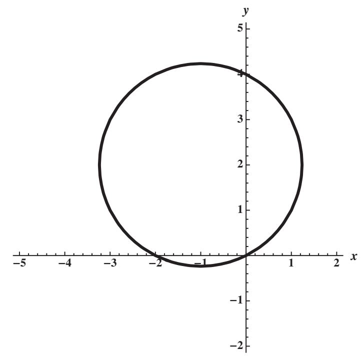
D1.3.27

$yy'(t) = t$, so $\int y\,dy = \int t\,dt$, so $y^2/2 = t^2/2 + C$. Because $y(1) = 2$, we have $2 = 1/2 + C$, so $C = 3/2$. So $y^2 = t^2 + 3$.
The solution corresponds to the upper portion of the curve.



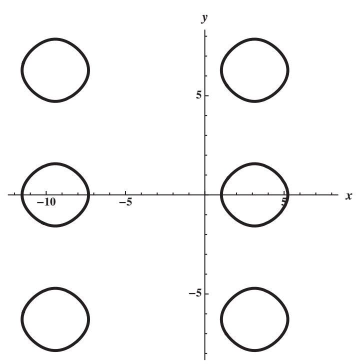
D1.3.28

$(2-y)y'(x) = 1+x$, so $\int(2-y)\,dy = \int(1+x)\,dx$, and $2y - y^2/2 = x + x^2/2 + C$. Because $y(1) = 1$, we have $2 - 1/2 = 1 + 1/2 + C$, so $C = 0$. Thus, $2y - y^2/2 = x + x^2/2$ describes the solution.



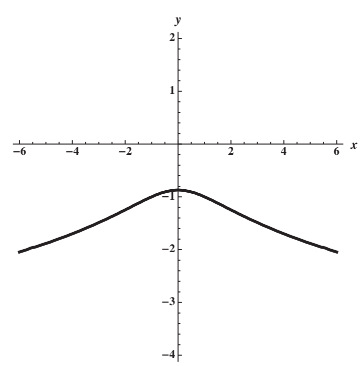
D1.3.29

$(\sin u)u'(x) = \cos(x/2)$, so $\int \sin u\,du = \int \cos(x/2)\,dx$. We have $-\cos u = 2\sin(x/2) + C$. Because $u(\pi) = \pi/2$, we have $-\cos\pi/2 = 2\sin(\pi/2) + C$, so $0 = 2(1) + C$, so $C = -2$. Thus, $-\cos u = 2\sin(x/2) - 2$ describes the solution. The curve in the middle in the rightmost column of curves is the particular one described by our initial condition, because it is the only one that contains the point $(\pi, \pi/2)$.



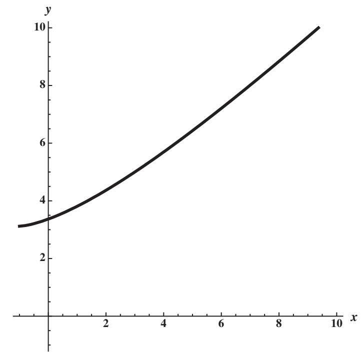
D1.3.30

$(2+y^2)^2 yy'(x) = 2x$, so $\int(2+y^2)^2 yy'(x)\,dx = \int 2x\,dx$, and thus $\frac{(2+y^2)^3}{6} = x^2 + C$. Because $y(1) = -1$, we have $C = \frac{21}{6}$. Thus, $(2+y^2)^3 = 6x^2 + 21$, and thus $y^2 = \sqrt[3]{6x^2+21} - 2$.



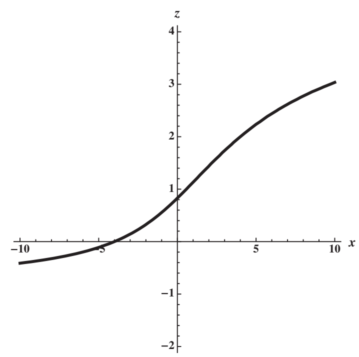
D1.3.31

$\sqrt{y+4}\,y'(x) = \sqrt{x+1}$, so $\int \sqrt{y+4}\,dy = \int \sqrt{x+1}\,dx$, so $\frac{2}{3}(y+4)^{3/2} = \frac{2}{3}(x+1)^{3/2} + C$. Because $y(3) = 5$, we have $\frac{2}{3}(27) = \frac{16}{3} + C$, so $C = \frac{38}{3}$. Thus, $\frac{2}{3}(y+4)^{3/2} = \frac{2}{3}(x+1)^{3/2} + \frac{38}{3}$, so $(y+4)^{3/2} = (x+1)^{3/2} + 19$



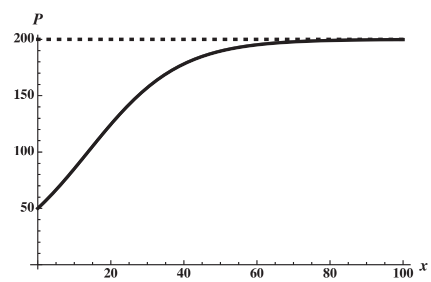
D1.3.32

$\frac{1}{z^2+4}z'(x) = \frac{1}{x^2+16}$, so $\int \frac{dz}{z^2+4} = \int \frac{dx}{x^2+16}$. Thus, $\frac{1}{2}\tan^{-1}(z/2) = \frac{1}{4}\tan^{-1}(x/4) + C$. Because $z(4) = 2$, we have $\pi/8 = \pi/16 + C$, so $C = \pi/16$. Thus, $2\tan^{-1}(z/2) = \tan^{-1}(x/4) + \pi/4$ describes the solution.



D1.3.33

a. This equation is separable, so we have $\int \frac{200}{P(200-P)}\,dP = \int 0.08\,dt$, so $\int \left(\frac{1}{P} + \frac{1}{200-P}\right)\,dP = 0.08t + C$. Therefore, $\ln\left|\frac{P}{200-P}\right| = 0.08t + C$. Substituting $P(0) = 50$ gives $-\ln 3 = C$, and solving for P gives $P(t) = \frac{200}{3e^{-0.08t}+1}$.

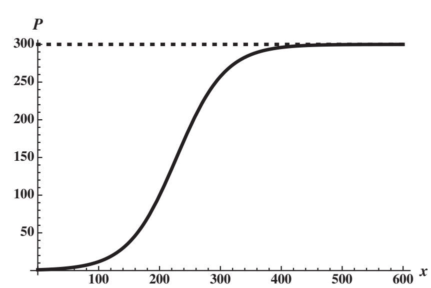


b. The steady-state population is $\lim_{t\to\infty} P(t) = 200$.

D1.3.34

a. This equation is separable, so we have $\int \frac{A}{P(A-P)}\,dP = \int k\,dt$, so $\int \left(\frac{1}{P} + \frac{1}{A-P}\right)\,dP = kt + D$. Therefore, $\ln\left|\frac{P}{A-P}\right| = kt + D$, which is equivalent to $\frac{P}{A-P} = Ce^{kt}$. Substituting $P(0) = P_0$ gives $C = P_0/(A-P_0)$, and solving for P gives $P(t) = \frac{AP_0}{P_0+(A-P_0)e^{-kt}}$.

b.



c. The denominator in $P(t)$ above is positive for all $t \geq 0$ when $0 < P_0 < A$, so $P(t)$ is defined for all $t \geq 0$; we have $\lim_{t\to\infty} P(t) = A$, which is the steady-state solution.

D1.3.35

a. True. It can be written as $u^7 u'(x) = x^{-2}$.

b. False.

c. True. When separated, we have $ye^y y'(x) = x$, and the left-hand side of the equation can be integrated by parts.

D1.3.36 We have $\int e^y y'(t)\,dt = \int \frac{\ln^2 t}{t}\,dt$, so $e^y = \frac{\ln^3 t}{3} + C$. Because $y(1) = \ln 2$, we have $2 = 0 + C$, so $e^y = \frac{\ln^3 t}{3} + 2$. Thus, $y = \ln\left(\frac{\ln^3 t+6}{3}\right)$.

D1.3.37 We have $\frac{1}{y(y+1)} y'(t) = \frac{3}{t}$, so $\int \frac{1}{y(y+1)}\,dy = \int \frac{3}{t}\,dt$, and thus $\ln\left|\frac{y}{y+1}\right| = 3\ln|t| + C$. Because $y(1) = 1$, we have $C = -\ln 2$. Thus, $\ln\left|\frac{y}{y+1}\right| = 3\ln|t| - \ln 2$ describes the solution. Then $\frac{y}{y+1} = t^3/2$, so $1 - \frac{1}{y+1} = t^3/2$, so $\frac{1}{y+1} = \frac{2-t^3}{2}$. Thus $y + 1 = \frac{2}{2-t^3}$, so $y = \frac{t^3}{2-t^3}$.

D1.3.38 $\int 2y\,dy = \int \cos^2 t\,dt$, so $y^2 = \frac{t}{2} + \frac{\sin 2t}{4} + C$. Because $y(0) = -2$, we have $4 = 0 + 0 + C$, so $C = 4$. Thus, $y = -\sqrt{\frac{t}{2} + \frac{\sin 2t}{4} + 4}$.

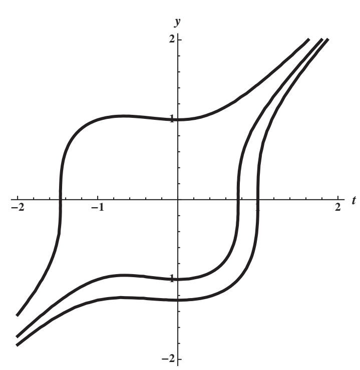
D1.3.39 Assume $y > -3$ and $t > -6/5$. $\int \left(\frac{1}{y+3}\right) y'(t)\,dt = \int \frac{1}{5t+6}\,dt$, so $\ln(y+3) = \frac{\ln(5t+6)}{5} + C$. We can write this as $5\ln(y+3) = \ln(5t+6) + 5C$, so $(y+3)^5 = A(5t+6)$. Because $y(2) = 0$, we have $3^5 = 16A$, so $A = 3^5/16$. We have $(y+3) = \frac{3\sqrt[5]{2}}{2}(5t+6)^{1/5}$. Thus, $y = -3 + \frac{3\sqrt[5]{2}}{2}(5t+6)^{1/5} = \frac{3}{2}(-2 + 2^{1/5}(6+5t)^{1/5})$.

D1.3.40

a. $\int y^2\,dy = \int (t^2 + 2t/3)\,dt$, so $\frac{y^3}{3} = t^3/3 + t^2/3 + C$. It is convenient to write this as $y^3 = t^3 + t^2 + C_1$.

b. When $y(-1) = 1$, we have $1 = -1 + 1 + C_1$, so $C_1 = 1$. When $y(1) = 0$, we have $0 = 1 + 1 + C_1$, so $C_1 = -2$. When $y(-1) = -1$, we have $-1 = -1 + 1 + C_1$, so $C_1 = -1$.

c.

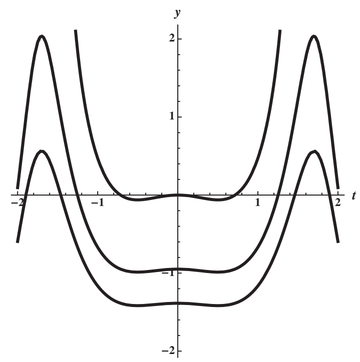


D1.3.41

a. $\int e^{-y/2}\,dy = \int (4x\sin x^2 - x)\,dx$. Thus, $-2e^{-y/2} = -2\cos x^2 - x^2/2 + K$. Then $e^{-y/2} = \cos x^2 + x^2/4 + C$, so $-y/2 = \ln(\cos x^2 + x^2/4 + C)$, and $y = -2\ln(\cos x^2 + x^2/4 + C)$.

b. When $y(0) = 0$ we have $0 = -2\ln(1 + C)$, so $C = 0$. When $y(0) = \ln(1/4)$, we have $\ln(1/4) = -2\ln(1+C)$, so $\ln 2 = \ln(1+C)$, so $C = 1$. When $y(\sqrt{\pi/2}) = 0$, we have $0 = -2\ln(0 + \pi/8 + C$, so $C = 1 - \frac{\pi}{8}$.

c.



D1.3.42

a. $4x + 2y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \frac{-4x}{2y} = \frac{-2x}{y}$.

b. Curves are orthogonal when their slopes are negative reciprocals of each other, and the negative reciprocal of $\frac{-2x}{y}$ is $\frac{y}{2x}$.

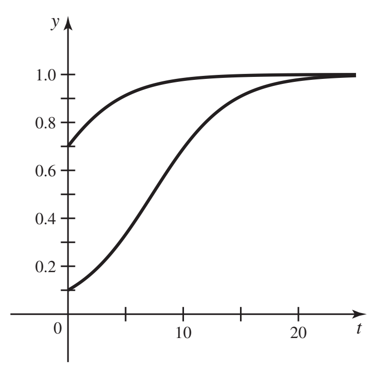
c. We have $\frac{2\,dy}{y} = \frac{dx}{x}$, so $2\ln|y| = \ln y^2 = \ln|x| + C$. Thus, $y^2 = e^C|x|$. We can write $\pm e^C = k$, so we have $y^2 = kx$.

D1.3.43 Differentiating implicitly gives $2x + 2yy' = 0$, so $y' = \frac{-x}{y}$. We are thus seeking curves so that $\frac{dy}{dx} = \frac{y}{x}$. We have $\frac{dy}{y} = \frac{dx}{x}$, so $\ln|y| = \ln|x| + C$ so $y = e^C|x| = kx$. So the family of curves we are seeking is the collection of curves $y = kx$.

D1.3.44

a. This equation is separable, so we have $\int \frac{1}{y(1-y)}\,dy = \int k\,dt$, so $\int \left(\frac{1}{y} + \frac{1}{1-y}\right)\,dy = kt + D$. Therefore, $\ln\left|\frac{y}{1-y}\right| = kt + D$, which is equivalent to $\frac{y}{1-y} = Ce^{kt}$. Substituting $y(0) = y_0$ gives $C = y_0/(1-y_0)$, and thus $y = \frac{y_0}{(1-y_0)e^{-kt}+y_0}$.

b.



c. The denominator in $y(t)$ above is positive for all $t \geq 0$ when $0 < y_0 < 1$, so $y(t)$ is defined for all $t \geq 0$; we have $\lim_{t\to\infty} y(t) = 1$, which is the steady-state solution.

D1.3.45

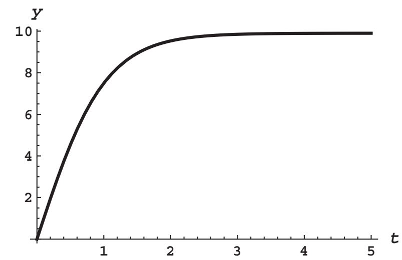
a. We have $mv'(t) = mg - kv^2$, so $v'(t) = g - av^2$ with $a = k/m$.

b. We solve $av^2 = g$ to obtain the terminal velocity $\tilde{v} = \sqrt{g/a} = \sqrt{gm/k}$.

c. This equation is separable, so we have $\int \frac{1}{g-av^2}dv = \int dt$, so $-\frac{1}{a}\int \frac{1}{v^2-\tilde{v}^2}\,dv = t + D$.

Thus, $-\frac{1}{2a\tilde{v}}\int\left(\frac{1}{v-\tilde{v}} - \frac{1}{v+\tilde{v}}\right)dv = t + D$, and $-\frac{1}{2a\tilde{v}}\ln\left|\frac{v-\tilde{v}}{v+\tilde{v}}\right| = t + D$, hence $\frac{v-\tilde{v}}{v+\tilde{v}} = Ce^{-2a\tilde{v}t}$ The initial condition $v(0) = 0$ gives $C = -1$, and solving for v gives $v = \frac{1-e^{-2a\tilde{v}t}}{1+e^{-2a\tilde{v}t}}\tilde{v}$. This can be written as $v = (\sqrt{g/a})\frac{e^{2\sqrt{ag}t}-1}{e^{2\sqrt{ag}t}+1}$.

d. We have $a = 0.1$, $\tilde{v} \approx 9.90$ m/s



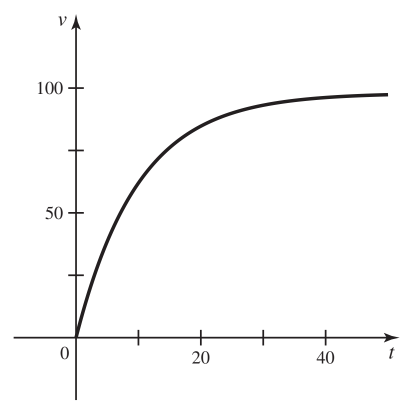
D1.3.46

a. We have $mv'(t) = mg - Rv$, so $v'(t) = g - bv$ with $b = R/m$.

b. Solve $bv = g$ to obtain terminal velocity $\tilde{v} = g/b = mg/R$.

c. The equation $v' = g - bv$ is first-order linear, with general solution $v = Ce^{-bv} + \tilde{v}$. The initial condition $v(0) = 0$ gives $C = -\tilde{v}$, which gives $v = \tilde{v}(1 - e^{-bt})$.

d. We have $b = 0.1$, $\tilde{v} = 98$ m/s.



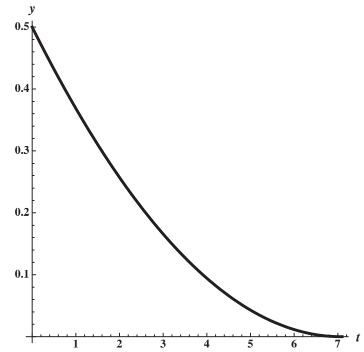
D1.3.47

a. The equation $h' = 2k\sqrt{h}$ is separable, so we have $\int \frac{dh}{2\sqrt{h}} = \int k\,dt$, so $\sqrt{h} = kt + C$. The initial condition $h(0) = H$ gives $C = \sqrt{H}$, so the solution is $h = (\sqrt{H} + kt)^2$.

b. The solution for $k = -0.1$ and $H = 0.5$ is $h = (0.7071 - 0.1t)^2$.

c. The tank is drained when $h(t) = 0$, which gives $t = -\sqrt{H}/k$. With $k = -.1$ and $H = .5$, we have $t = \sqrt{.5}/.1 \approx 7.07$ seconds.

d.

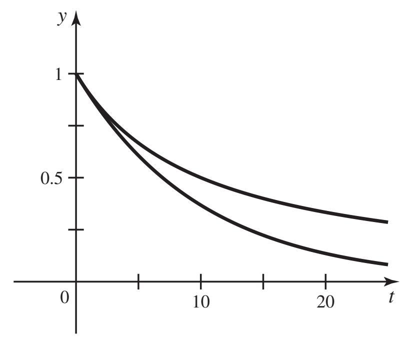


D1.3.48

a. The general solution to $y' = -ky$ is $y = Ce^{-kt}$.

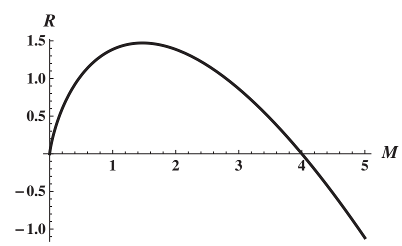
b. The equation $y' = -ky^2$ is separable, so we have $-\int \frac{dy}{y^2} = \int k\,dt$, so $\frac{1}{y} = kt + C$. The initial condition $y(0) = y_0$ gives $C = 1/y_0$, and solving for y gives $y = \frac{1}{kt+1/y_0} = \frac{y_0}{1+ky_0t}$.

c.



D1.3.49

a. The growth rate is positive when $0 < M < 4$. The function $R(M)$ has derivative $R'(M) = -r\left(\ln\left(\frac{M}{K}\right) + M \cdot \frac{K}{M} \cdot \frac{1}{K}\right) = -r\left(\ln\left(\frac{M}{K}\right) + 1\right)$ which is 0 when $M/4 = 1/e$ or $M = 4/e$. We also observe that $\lim_{M\to 0^+} R(M) = 0$ and $R(4) = 0$, so $R(M)$ takes its maximum at the critical point $M = 4/e$.



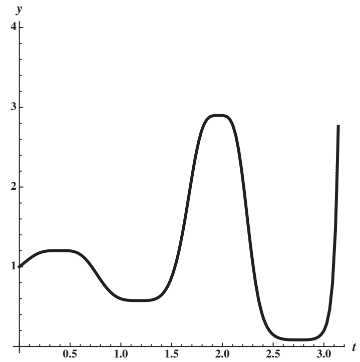
b. The equation is separable, so we have $\int \frac{dM}{M(\ln M - \ln K)} = -\int r\,dt$, so $\ln|\ln M - \ln K| = -rt + D$, and thus $\ln\left(\frac{M}{K}\right) = Ce^{-rt}$. Therefore $M = Ke^{Ce^{-rt}}$.

The conditions $r = 1$, $K = 4$ and $M_0 = 1$ give $C = -\ln 4$ and $M = 4e^{(-\ln 4)e^{-t}} = 4^{1-e^{-t}}$. Observe that $\lim_{t\to\infty} M(t) = 4e^0 = 4$, so the limiting size of the tumor is 4.

c. In general, the limiting size of the tumor is $\lim_{t\to\infty} Ke^{Ce^{-rt}} = K$, because $r > 0$.

D1.3.50 Solving with a CAS gives

$$y = e^{\left(\frac{e^t(1740\sin(4t)+204\sin(12t)+435\cos(4t)+17\cos(12t))}{9860} - \frac{113}{2465}\right)}.$$



D1.3.51

a. $\frac{1}{y^2}y'(t) = 1$, so $\int \frac{dy}{y^2} = \int 1\,dt$. Thus $\frac{-1}{y} = t + C$. Because $y_0 = 1$ we have $-1 = C$. Thus, $y = \frac{1}{1-t}$.

b. $\frac{1}{y^3}y'(t) = 1$, so $\int \frac{dy}{y^3} = \int 1\,dt$. Thus $\frac{-1}{2y^2} = t + C$. Because $y_0 = \frac{1}{\sqrt{2}}$ we have $-1 = C$. Thus, $\frac{1}{2y^2} = 1 - t$ and $y = \frac{1}{\sqrt{2}\sqrt{1-t}}$.

c. $\frac{1}{y^{n+1}}y'(t) = 1$, so $\int \frac{dy}{y^{n+1}} = \int 1\,dt$. Thus $\frac{-1}{ny^n} = t + C$. Because $y_0 = n^{-1/n}$ we have $-1 = C$. Thus, $\frac{1}{ny^n} = 1 - t$, and $ny^n = \frac{1}{1-t}$. Thus, $y = \frac{1}{(n(1-t))^{1/n}}$.

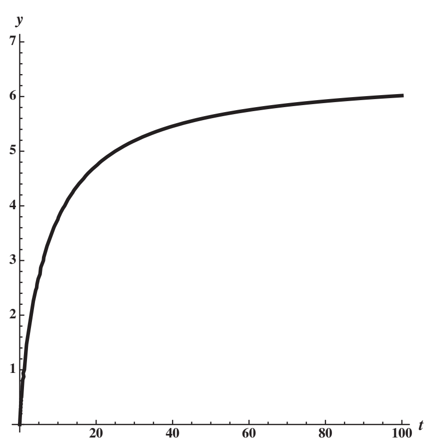
We have $\lim_{t\to 1^-} \frac{1}{(n(1-t))^{1/n}} = \infty$.

D1.3.52

a. $\frac{1}{y(y+1)}y'(t) = \frac{1}{t(t+2)}$, so $\int \frac{dy}{y(y+1)} = \int \frac{dt}{t(t+2)}$. Using partial fraction decompositions, we have $\ln\left|\frac{y}{y+1}\right| = \frac{1}{2}\ln\left|\frac{t}{t+2}\right| + K$. Thus $\left|\frac{y}{y+1}\right| = D\sqrt{\frac{t}{t+2}}$. Thus, $1 - \frac{1}{y+1} = \pm D\sqrt{\frac{t}{t+2}}$, and thus $1 \mp D\sqrt{\frac{t}{t+2}} = \frac{1}{y+1}$, so $y + 1 = \frac{1}{1\mp D\sqrt{\frac{t}{t+2}}}$, and $y = \frac{1}{1\mp D\sqrt{\frac{t}{t+2}}} - 1 = \frac{\pm D\sqrt{\frac{t}{t+2}}}{1\mp D\sqrt{\frac{t}{t+2}}}$. This can be written as $\frac{\pm D\sqrt{\frac{t}{t+2}}}{1\mp D\sqrt{\frac{t}{t+2}}} \cdot \frac{\sqrt{t+2}}{\sqrt{t+2}} \cdot \frac{\frac{1}{\pm D}}{\frac{1}{\pm D}} = \frac{\sqrt{t}}{C\sqrt{t+2}-\sqrt{t}}$ where $C = \frac{1}{\pm D}$.

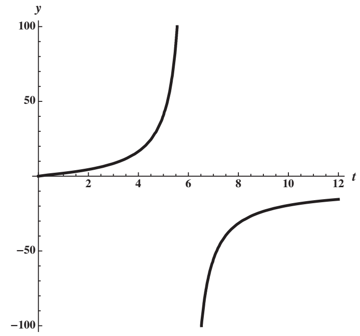
b. If $y(1) = A$, then $A = \frac{1}{\sqrt{3}C-1}$, so $C = \frac{1+A}{\sqrt{3}A}$.

c. If $A = 1$, then $C = \frac{2}{\sqrt{3}}$, so the solution is $y = \frac{\sqrt{t}}{\frac{2\sqrt{t+2}}{\sqrt{3}} - \sqrt{t}}$.



d. $\lim_{t\to\infty} y(t) = \frac{1}{\frac{2}{\sqrt{3}}-1} \approx 6.46$.

e. When $y(1) = 2$, we have $C = \frac{\sqrt{3}}{2}$, so the solution is $y = \frac{\sqrt{t}}{\frac{\sqrt{3}\sqrt{t+2}}{2} - \sqrt{t}}$.



f. $\lim_{t\to\infty} y(t) = \frac{1}{\frac{\sqrt{3}}{2}-1} \approx -7.46$.

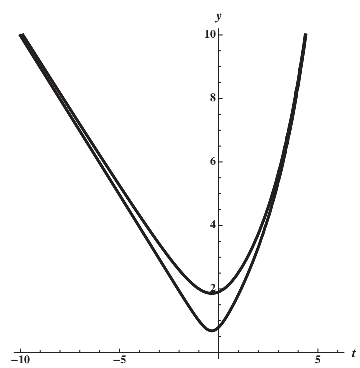
g. $\lim_{t\to\infty} y(t) = \frac{1}{C-1} = \frac{\sqrt{3}A}{1+A(1-\sqrt{3})}$.

D1.3.53

a. $\int yy'(t)\,dt = \int (e^t/2 + t)\,dt$, so $y^2/2 = e^t/2 + t^2/2 + C$, so $y^2 = e^t + t^2 + C_1$. So $y = \pm\sqrt{e^t + t^2 + C_1}$.

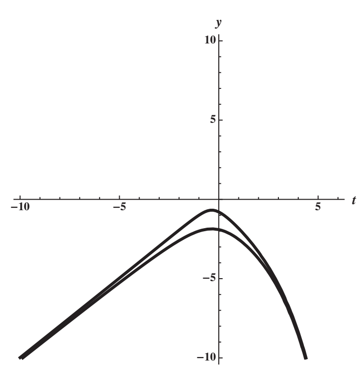
b. If $y(-1) = 1$, then $1 = \sqrt{1/e + 1 + C_1}$ so $C_1 = -1/e$, so $y = \sqrt{e^t + t^2 - 1/e}$.
If $y(-1) = 2$, then $2 = \sqrt{1/e + 1 + C_1}$ so $C_1 = 3 - 1/e$, so $y = \sqrt{e^t + t^2 + 3 - 1/e}$.

c. For $t > 0$, the solutions increase as t increases.



d. If $y(-1) = -1$, then $-1 = -\sqrt{1/e + 1 + C_1}$, so $C_1 = -1/e$ and $y = -\sqrt{e^t + t^2 - 1/e}$.
If $y(-1) = -2$, then $-2 = -\sqrt{1/e + 1 + C_1}$ so $C_1 = 3 - 1/e$, so $y = -\sqrt{e^t + t^2 + 3 - 1/e}$.

e. For $t > 0$, the solutions decrease as t increases.



D1.4 Special First-Order Linear Differential Equations

D1.4.1 Because $y(0) = 4$, we have $4 = C - 13$, so $C = 17$. Thus, the solution is $y = 17e^{-10t} - 13$.

D1.4.2 The general solution is $y = Ce^{3t} - \frac{-12}{3} = Ce^{3t} + 4$.

D1.4.3 The general solution is $y = Ce^{-4t} - \frac{6}{-4} = Ce^{-4t} + \frac{3}{2}$.

D1.4.4 The equilibrium solution is $y = 3$. It is unstable.

D1.4.5 Because $k = 3$ and $b = -4$, we have $y = Ce^{3t} + \frac{4}{3}$.

D1.4.6 Because $k = -1$ and $b = 2$, we have $y = Ce^{-x} + 2$.

D1.4.7 Because $k = -2$ and $b = -4$, we have $y = Ce^{-2x} - 2$.

D1.4.8 Because $k = 2$ and $b = 6$, we have $y = Ce^{2x} - 3$.

D1.4.9 Because $k = -12$ and $b = 15$, we have $u = Ce^{-12t} + \frac{5}{4}$.

D1.4.10 Because $k = \frac{1}{2}$ and $b = 14$, we have $v = Ce^{y/2} - 28$.

D1.4.11 Because $k = 3$ and $b = -6$, we have $y = Ce^{3t} + 2$. Because $y(0) = 9$, we have $9 = C + 2$, so $C = 7$. Thus, the solution is $y = 7e^{3t} + 2$.

D1.4.12 Because $k = -1$ and $b = 2$, we have $y = Ce^{-x} + 2$. Because $y(0) = -2$, we have $-2 = C + 2$, so $C = -4$. Thus, the solution is $y = -4e^{-x} + 2$.

D1.4.13 Because $k = 2$ and $b = 8$, we have $y = Ce^{2t} - 4$. Because $y(0) = 0$, we have $0 = C - 4$, so $C = 4$. Thus, the solution is $y = 4e^{2t} - 4$.

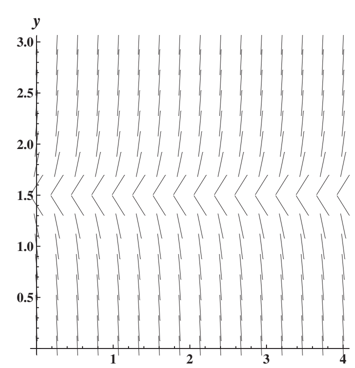
D1.4.14 Because $k = 2$ and $b = 6$, we have $u = Ce^{2x} - 3$. Because $u(1) = 6$, we have $6 = Ce^{2} - 3$, so $C = 9e^{-2}$. Thus, the solution is $u = 9e^{2x-2} - 3$.

D1.4.15 Because $k = 3$ and $b = 12$, we have $y = Ce^{3t} - 4$. Because $y(1) = 4$, we have $4 = Ce^{3} - 4$, so $C = 8e^{-3}$. Thus, the solution is $y = 8e^{3t-3} - 4$.

D1.4.16 Because $k = -1/2$ and $b = 6$, we have $z = Ce^{-t/2} + 12$. Because $z(-1) = 0$, we have $0 = Ce^{1/2} + 12$, so $C = -12e^{-1/2}$. Thus, the solution is $z = -12e^{-t/2-1/2} + 12$.

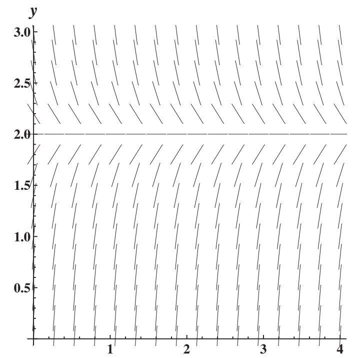
D1.4.17

The equilibrium solution is $y = \frac{18}{12} = \frac{3}{2}$. The solution is unstable.



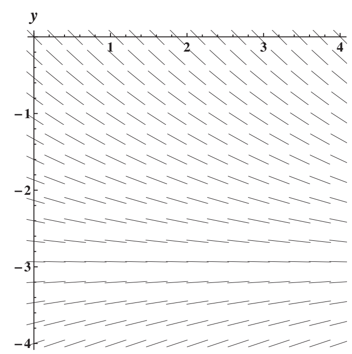
D1.4.18

The equilibrium solution is $y = 2$. The solution is stable.



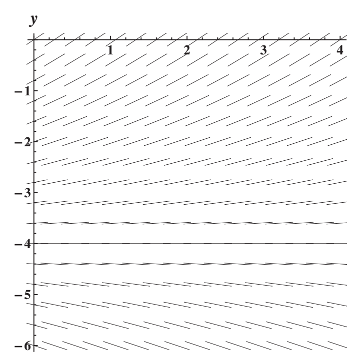
D1.4.19

The equilibrium solution is $y = -3$. The solution is stable.



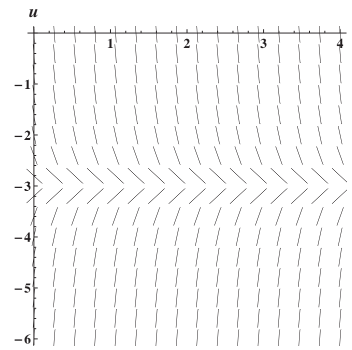
D1.4.20

The equilibrium solution is $y = -4$. The solution is unstable.



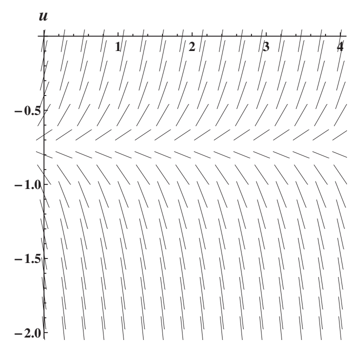
D1.4.21

The equilibrium solution is $u = -3$. The solution is stable.



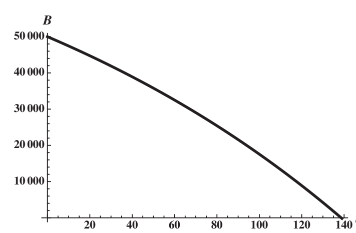
D1.4.22

The equilibrium solution is $u = -3/4$. The solution is unstable.



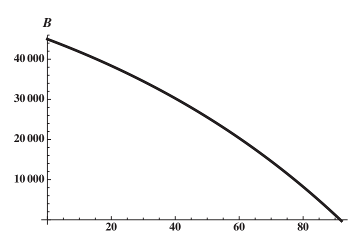
D1.4.23

Because $k = .005$ and $b = -500$, we have $B = Ce^{.005t} + \frac{500}{.005} = Ce^{.005t} + 100000$. Because $B(0) = 50,000$, we have $50000 = C + 100000$, so $C = -50000$. Thus, $B = -50000e^{.005t} + 100000$.
The balance is zero when $t = \ln(2)/.005 \approx 139$ months.



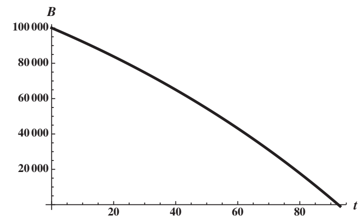
D1.4.24

Because $k = .01$ and $b = -750$, we have $B = Ce^{.01t} + \frac{750}{.01} = Ce^{.01t} + 75000$. Because $B(0) = 45,000$, we have $45000 = C + 75000$, so $C = -30000$. Thus, $B = -30000e^{.01t} + 75000$.
The balance is zero when $t = \ln(5/2)/.01 \approx 92$ months.



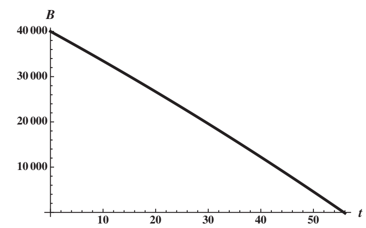
D1.4.25

Because $k = .0075$ and $b = -1500$, we have $B = Ce^{.0075t} + \frac{1500}{.0075} = Ce^{.0075t} + 200000$. Because $B(0) = 100,000$, we have $100000 = C + 200000$, so $C = -100000$. Thus, $B = -100000e^{.0075t} + 200000$.
The balance is zero when $t = \ln(2)/.0075 \approx 93$ months.



D1.4.26

Because $k = .004$ and $b = -800$, we have $B = Ce^{.004t} + \frac{800}{.004} = Ce^{.004t} + 200000$. Because $B(0) = 40,000$, we have $40000 = C + 200000$, so $C = -160000$. Thus, $B = -160000e^{.004t} + 200000$.
The balance is zero when $t = \ln(5/4)/.004 \approx 56$ months.



D1.4.27 We know that the solution has the form $T(t) = (90 - 25)e^{-kt} + 25 = 65e^{-kt} + 25$. Because $T(1) = 85$, we have $65e^{-k} + 25 = 85$. Thus, $k = -\ln(60/65) \approx .08$. Thus, $T(t) = 65e^{-.08t} + 25$. We have $T(t) = 30$ when $65e^{-.08t} + 25 = 30$, or when $e^{-.08t} = \frac{1}{13}$. This occurs when $t = \ln(13)/.08 \approx 32$ minutes after the coffee is first poured.

D1.4.28 We have $T(t) = (900 - 30)e^{-.02t} + 30$. We have $T(t) = 100$ when $870e^{-.02t} + 30 = 100$, which occurs when $e^{-.02t} = \frac{7}{87}$. This occurs when $t = \frac{\ln(7/87)}{-.02} \approx 126$.

D1.4.29 $T(t) = (5 - 20)e^{-kt} + 20 = -15e^{-kt} + 20$. Because $T(1) = 7$, we have $-15e^{-k} + 20 = 7$, so $k = \ln(15/13) \approx .143$. Thus $T(t) = -15e^{-.143t} + 20$. The milk will reach 18 degrees when $-15e^{-.143t} + 20 = 18$, which occurs when $e^{-.143t} = 2/15$. Thus, when $t \approx \frac{\ln(2/15)}{-.143} \approx 14$ minutes.

D1.4.30 $T(t) = (100 - 10)e^{-kt} + 10 = 90e^{-kt} + 10$. We are given that $T(30) = 80$, so $80 = 90e^{-30k} + 10$, so $e^{-30k} = 7/9$. Thus $k = \frac{\ln(7/9)}{-30} \approx .0084$. The soup will reach 30 degrees when $30 = 90e^{-.0084t} + 10$, which occurs when $e^{-.0084t} = 2/9$, or $t = \frac{\ln(2/9)}{-.0084} \approx 179$ minutes.

D1.4.31

a. False. It is $y(t) = Ce^{2t} + 9$ where C is an arbitrary constant.

b. True. Note that if $y(t) = 0$, then $y'(t) = 0$, not $k(0) - b = -b$ as dictated by the equation.

c. False. It is not separable.

d. False. It approaches it asymptotically.

D1.4.32 First note that if $ky + b < 0$, then $|ky + b| = -ky - b$. We will use this fact in what follows. We have $\frac{1}{ky+b}y'(t) = 1$, so $\int \frac{dy}{ky+b} = \int dt$, so $\frac{1}{k}\ln(-ky - b) = t + C_1$. Thus $\ln(-ky - b) = kt + C_2$, so $-ky - b = C_3e^{kt}$, and $y = Ce^{-kt} - \frac{b}{k}$.

D1.4.33 $\int (ty'(t) + y(t))\,dt = \int (1 + t)\,dt$, so $ty(t) = t + t^2/2 + C$. Thus, $y(t) = 1 + t/2 + C/t$. Because $y(1) = 4$, we have $4 = 1 + 1/2 + C$, so $C = 5/2$. Thus, $y(t) = 1 + t/2 + 5/(2t)$.

D1.4.34 $\int (t^3y'(t) + 3t^2y(t))\,dt = \int (1/t + 1)\,dt$. Thus, $t^3y(t) = \ln|t| + t + C$. Therefore, $y(t) = \frac{\ln|t|}{t^3} + \frac{1}{t^2} + \frac{C}{t^3}$. Because $y(1) = 6$, we have $6 = 0 + 1 + C$, so $C = 5$. The solution is thus $y(t) = \frac{\ln|t|}{t^3} + \frac{1}{t^2} + \frac{5}{t^3}$.

D1.4.35 $\int (e^{-t}y'(t) - e^{-t}y(t))\,dt = \int e^{2t}\,dt$, so $e^{-t}y(t) = e^{2t}/2 + C$. Thus, $y(t) = e^{3t}/2 + Ce^t$. Because $y(0) = 4$, we have $4 = 1/2 + C$, so $C = 7/2$. The solution is therefore $y(t) = e^{3t}/2 + 7e^t/2$.

D1.4.36 $\int ((t^2+1)y'(t) + 2ty(t))\,dt = \int 3t^2\,dt$, so $(t^2+1)y(t) = t^3 + C$. Thus, $y(t) = \frac{t^3}{t^2+1} + \frac{C}{t^2+1}$. Because $y(2) = 8$, we have $8 = 8/5 + C/5$, so $C = 32$. The solution is therefore $y(t) = \frac{t^3+32}{t^2+1}$.

D1.4.37

a. Note that $k = .03$ and $b = -600$. Thus, $B(t) = Ce^{.03t} + 20,000$. If $B_0 = B(0)$ is the amount borrowed, then $B_0 = C + 20,000$, so $C = B_0 - 20,000 = 20,000$.

Thus, $B(t) = 20,000e^{.03t} + 20,000$. This is an increasing function because its derivative is positive. It is occurring because the amount paid monthly is less than the monthly accruing interest.

b. Because $20000 \cdot .03 = 600$, the amount borrowed should be less than 20000 if the balance is to be decreasing.

c. The maximum amount that can be borrowed and not have the unpaid balance increase is B_0 when $rB_0 - m = 0$, or $B_0 = \frac{m}{r}$.

D1.4.38

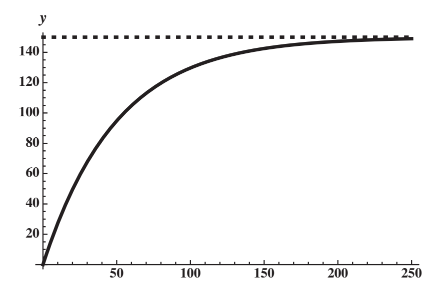
a. $T(t) = (T_0 - A)e^{-kt} + A$. We are seeking $t_{1/2}$ so that $T(t_{1/2}) = \frac{T_0}{2}$. Thus we have $T_0/2 - A = (T_0 - A)e^{-kt_{1/2}}$, so $\frac{T_0 - 2A}{2(T_0 - A)} = e^{-kt_{1/2}}$. Thus, $t_{1/2} = \frac{-1}{k} \ln \left| \frac{T_0 - 2A}{2(T_0 - A)} \right|$.

b. As k increases, $t_{1/2}$ decreases.

c. If $A > T_0/2$, then the equation $T_0/2 - A = (T_0 - A)e^{-kt_{1/2}}$ would have no solution, because the left-hand side is negative and the right-hand side is positive.

D1.4.39

a. The general solution is $y(t) = Ce^{-0.02t} + 150$; substitute $y(0) = 0$ to obtain $C + 150 = 0$, so $C = -150$; hence $y(t) = 150(1 - e^{-0.02t})$.



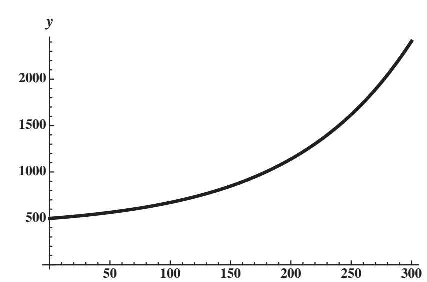
b. The steady-state level is $\lim_{t\to\infty} 150(1 - e^{-0.02t}) = 150\,\text{mg}$.

c. We have $150(1 - e^{-0.02t}) = 0.9 \cdot 150$, so $e^{-0.02t} = 0.1$, and thus $t = \frac{\ln 10}{0.02} \approx 115.1$ hours.

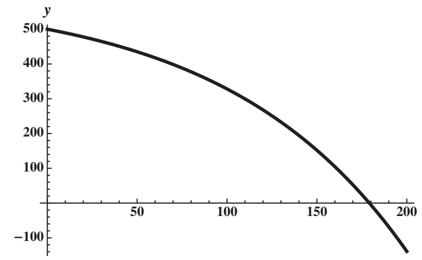
D1.4.40

a. The general solution is $y(t) = Ce^{0.01t} + 10b$; substitute $y(0) = 500$ to obtain $C + 10b = 500$, so $C = 500 - 10b$; hence $y(t) = (500 - 10b)e^{0.01t} + 10b$.

b. In this case the solution is $y(t) = 100e^{0.01t} + 400$, which approaches ∞ as $t \to \infty$.



c. In this case the solution is $y(t) = -100e^{0.01t} + 600$, which reaches 0 when $e^{0.01t} = 6$ or $t = 100 \ln 6 \approx 179$ yrs.



D1.4.41

a. The equation $y'(t) = 0.008y - h$ has steady-state solution $y = h/0.008 = 125h$, so we solve $y_0 = 2000 = 125h$ to obtain $h = 16$.

b. If $h = 200$, then the steady-state solution is $y = 125 \cdot 200 = 25{,}000$.

D1.4.42

a. The equation $B' = rB - m$ is first-order linear, with general solution $B = Ce^{rt} + m/r$; in this case $r = 0.05$, $m/r = 20{,}000$, and the initial condition $B_0 = 15{,}000$ gives $C = -5000$, so $B = 20{,}000 - 5000e^{0.05t}$. The balance decreases.

b. The steady-state (constant balance) solution is $B = m/r = \$50{,}000$, which gives $m = 0.05 \cdot 50{,}000 = \$2500$.

D1.4.43

a. Let $v = y^{1-p}$. Then $v'(t) = (1-p)y^{-p}y'(t)$, so $y'(t) = \frac{y^p}{1-p}v'(t)$.

b. Given $y'(t) + ay = by^p$, we have $\frac{y^p}{1-p}v'(t) + ay = by^p$, so $\frac{1}{1-p}v'(t) + ay^{1-p} = b$, so

$$v'(t) = (p-1)av(t) + b(1-p).$$

Then $v(t) = Ce^{(p-1)at} - \frac{b(1-p)}{(p-1)a} = Ce^{(p-1)at} + \frac{b}{a}$. Therefore,

$$y(t) = \left(Ce^{(p-1)at} + \frac{b}{a}\right)^{1/(1-p)}.$$

D1.4.44

a. Using the results of the previous problem, we have $a = 1$, $b = 2$, and $p = 2$. Thus

$$y(t) = \left(Ce^t + 2\right)^{-1}.$$

b. We have $a = -2$, $p = -1$, and $b = 3$, so

$$y(t) = \left(Ce^{4t} + \frac{-3}{2}\right)^{1/2}.$$

c. We have $a = 1$, $b = 1$, and $p = 1/2$, so

$$y(t) = \left(Ce^{-t/2} + 1\right)^2.$$

D1.4.45 Let $p(t) = \exp(\int 1/t\,dt) = e^{\ln t} = t$. The original differential equation can be written as $t(y'(t) + (1/t)y(t)) = 0$, or $ty'(t) + y(t) = 0$. Integrating both sides with respect to t gives $ty(t) = C$, so $y(t) = \frac{C}{t}$. Because $y(1) = 6$, we have $C = 6$,and $y = \frac{6}{t}$.

D1.4.46 Let $p(t) = \exp(\int 3/t\,dt) = e^{3\ln t} = t^3$. The original differential equation can be written as $t^3(y'(t)+(3/t)y(t)) = t^3 - 2t^4$, or $t^3y'(t) + 3t^2y(t) = t^3 - 2t^4$. Integrating both sides with respect to t gives $t^3y(t) = t^4/4 - 2t^5/5 + C$, so $y(t) = t/4 - 2t^2/5 + C/t^3$. Because $y(2) = 0$, we have $0 = \frac{1}{2} - \frac{8}{5} + \frac{C}{8}$, and solving for C gives $C = \frac{44}{5}$. Thus, $y(t) = \frac{t}{4} - \frac{2t^2}{5} + \frac{44}{5t^3}$.

D1.4.47 Let $p(t) = \exp(\int \frac{2t}{t^2+1}\,dt) = \exp(\ln(t^2 + 1)) = t^2 + 1$. The original differential equation can be written as $(t^2 + 1)y'(t) + (2t)y(t) = (t^2 + 1)(1 + 3t^2)$. Integrating both sides with respect to t gives $(t^2 + 1)y(t) = \int(3t^4 + 4t^2 + 1)\,dt = 3t^5/5 + 4t^3/3 + t + C$. Because $y(1) = 4$, we have $8 = 3/5 + 4/3 + 1 + C$, so $C = \frac{76}{15}$. Thus, $y(t) = \frac{3t^5/5+4t^3/3+t+\frac{76}{15}}{t^2+1} = \frac{9t^5+20t^3+15t+76}{15(t^2+1)}$.

D1.4.48 Let $p(t) = \exp(\int 2t\,dt) = e^{t^2}$. The original differential equation can be written as $e^{t^2}y'(t) + e^{t^2}(2t)y(t) = 3te^{t^2}$. Integrating both sides with respect to t gives $e^{t^2}y(t) = \frac{3}{2}e^{t^2} + C$. Because $y(0) = 1$, we have $1 = \frac{3}{2} + C$, so $C = \frac{-1}{2}$. Thus, $y(t) = \frac{\frac{3}{2}e^{t^2} - \frac{1}{2}}{e^{t^2}} = \frac{3e^{t^2}-1}{2e^{t^2}}$.

D1.5 Modeling with Differential Equations

D1.5.1 The growth rate function specifies the rate of growth of the population. If the growth rate function is positive, then the population is increasing, while the population is decreasing when the growth rate function is negative.

D1.5.2 The carrying capacity is the upper limit of the size of a population, due to limitations in resources. Mathematically, it appears as a horizontal asymptote as $t \to \infty$.

D1.5.3 If the growth rate function is positive and decreasing, then the population is increasing. Whether or not the population is increasing is completely determined by whether the growth rate function is positive or negative.

D1.5.4 A stirred tank reaction takes place in a tank that is filled with a soluble substance like salt or sugar. The tank has an inflow and an outflow pipe, so the volume of the solution is constant. The tank is assumed to be completely stirred at all times, so the solution is uniform. The problem is to find the mass of the substance in the tank at all times.

D1.5.5 Is is a linear, first-order differential equation.

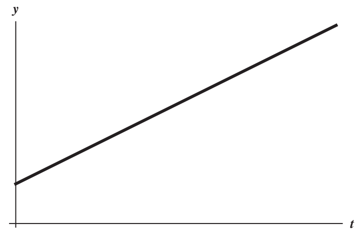
D1.5.6

- In the absence of prey, the predator population deceases exponentially, while encounters between prey and predators increase the predator population (the prey are the food supply).
- In the absence of predators, the prey population increases exponentially, while encounters between the prey and predators deplete the prey population (the predators eat the prey).

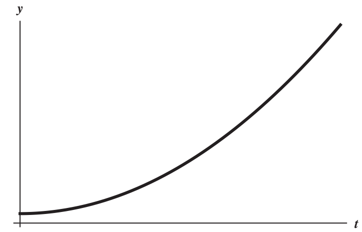
D1.5.7 The solution curves in the FH-plane are closed curves that circulate around the equilibrium point.

D1.5.8 They both oscillate cyclically, with the prey population peaking slightly before the predator population.

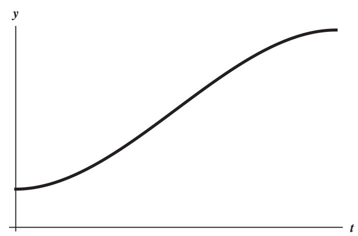
D1.5.9



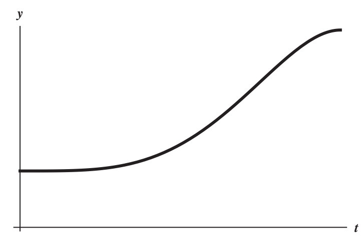
D1.5.10



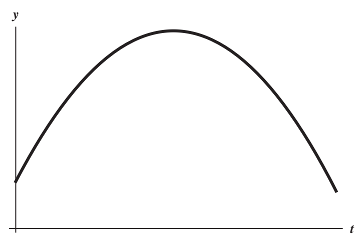
D1.5.11



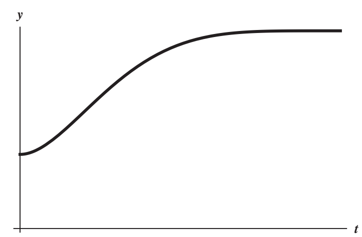
D1.5.12



D1.5.13

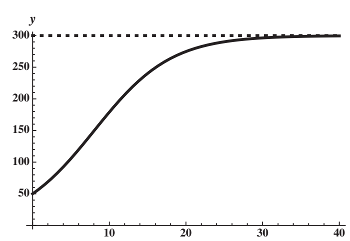


D1.5.14



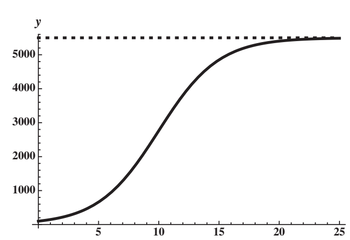
D1.5.15

We must solve $P'(t) = .2P\left(1 - \frac{P}{300}\right)$. We have $\int \frac{300P'(t)}{P(300-P)}\,dt = \int .2\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{300-P(t)}\right)dt = \int .2\,dt$. Thus, $\ln\left(\frac{P(t)}{300-P(t)}\right) = .2t + C$. Taking the exponential of both sides and reciprocating gives $\frac{300-P(t)}{P(t)} = Ae^{-.2t}$. Because $P(0) = 50$, we have $A = 5$. Thus $\frac{300}{P(t)} = 5e^{-.2t} + 1$, so $P(t) = \frac{300}{5e^{-.2t}+1}$.



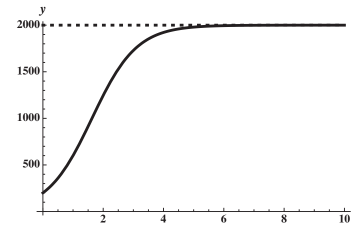
D1.5.16

We must solve $P'(t) = .4P\left(1 - \frac{P}{5500}\right)$. We have $\int \frac{5500P'(t)}{P(5500-P)}\,dt = \int .4\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{5500-P(t)}\right)dt = \int .4\,dt$. Thus, $\ln\left(\frac{P(t)}{5500-P(t)}\right) = .4t + C$. Taking the exponential of both sides and reciprocating gives $\frac{5500-P(t)}{P(t)} = Ae^{-.4t}$. Because $P(0) = 100$, we have $A = 54$. Thus $\frac{5500}{P(t)} = 54e^{-.4t} + 1$, so $P(t) = \frac{5500}{54e^{-.4t}+1}$.



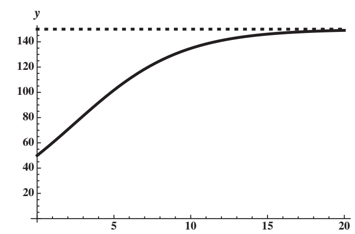
D1.5.17

We must solve $P'(t) = rP\left(1 - \frac{P}{2000}\right)$. We have $\int \frac{2000P'(t)}{P(2000-P)}\,dt = \int r\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{2000-P(t)}\right)dt = \int r\,dt$. Thus, $\ln\left(\frac{P(t)}{2000-P(t)}\right) = rt + C$. Taking the exponential of both sides and reciprocating gives $\frac{2000-P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = 200$, we have $A = 9$. Thus $\frac{2000}{P(t)} = 9e^{-rt}+1$, so $P(t) = \frac{2000}{9e^{-rt}+1}$. Now because $P(1) = 600$, we have $r = \ln(27/7)$. So $P(t) = \frac{2000}{9e^{\ln(7/27)t}+1}$.



D1.5.18

We must solve $P'(t) = rP\left(1 - \frac{P}{150}\right)$. We have $\int \frac{150P'(t)}{P(150-P)}\,dt = \int r\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{150-P(t)}\right)dt = \int r\,dt$. Thus, $\ln\left(\frac{P(t)}{150-P(t)}\right) = rt+C$. Taking the exponential of both sides and reciprocating gives $\frac{150-P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = 50$, we have $A = 2$. Thus $\frac{150}{P(t)} = 2e^{-rt}+1$, so $P(t) = \frac{150}{2e^{-rt}+1}$. Now because $P(1) = 60$, we have $r = \ln(4/3)$. So $P(t) = \frac{150}{2e^{\ln(3/4)t}+1}$.

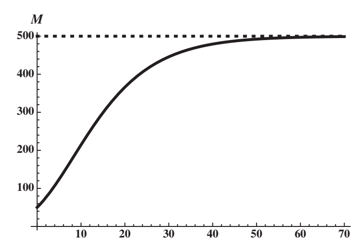


D1.5.19 We have $\frac{M'(t)}{M\ln(M/K)} = -r$, so $\int \frac{M'(t)}{M\ln(M/K)}\,dt = \int -r\,dt$.
Note that $\frac{d}{dt}\ln(M/K) = (K/M)(1/K)M'(t) = M'(t)/M$. Thus integrating gives $\ln|\ln(M/K)| = -rt + C$, and thus $\ln(M/K) = Ae^{-rt}$. Because $M(0) = M_0$, we have $A = \ln(M_0/K)$.
Thus $M(t) = K(\exp(\ln(M_0/K)e^{-rt})) = K\left(\frac{M_0}{K}\right)^{e^{-rt}}$.

D1.5.20

With $r = .1$, $K = 500$, and $M_0 = 50$, we have

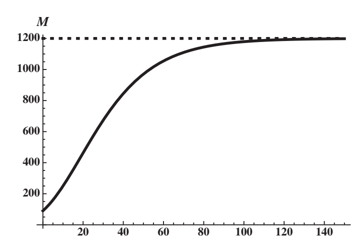
$$M(t) = 500\left(\frac{1}{10}\right)^{e^{-.1t}}.$$



D1.5.21

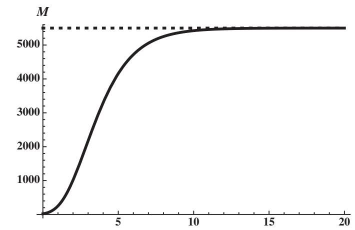
With $r = .05$, $K = 1200$, and $M_0 = 90$, we have

$$M(t) = 1200\left(\frac{3}{40}\right)^{e^{-.05t}}.$$



D1.5.22

With $r = .6$, $K = 5500$, and $M_0 = 20$, we have $M(t) = 5500\left(\frac{1}{275}\right)^{e^{-.6t}}$.

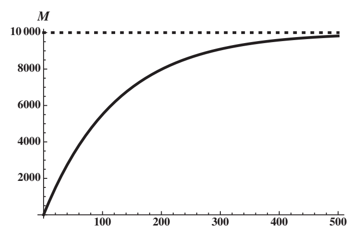


D1.5.23

a. The initial mass of copper sulfate is $m_0 = 0$. We have $m'(t) = -\frac{4}{500}m(t) + 20 \cdot 4 = \frac{-1}{125}m(t) + 80$.

b.

This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-.008t} + 10,000$. Because $m(0) = 0$, we have $C = -10,000$. Thus, $m(t) = -10,000e^{-.008t} + 10,000$.

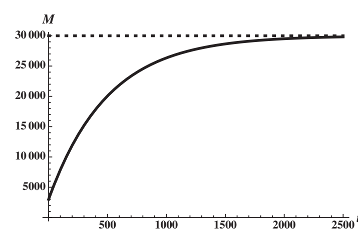


D1.5.24

a. The initial mass of the salt is $m_0 = 3000$. We have $m'(t) = -\frac{3}{1500}m(t) + 20 \cdot 3 = \frac{-1}{500}m(t) + 60$.

b.

This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-.002t} + 30,000$. Because $m(0) = 3000$, we have $C = -27,000$. Thus, $m(t) = -27,000e^{-.002t} + 30,000$.

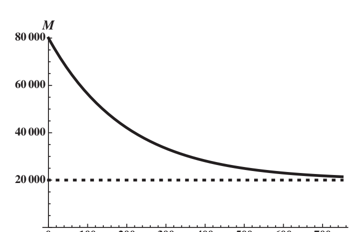


D1.5.25

a. The initial mass of the sugar is $m_0 = 2000 \cdot 40 = 80,000$. We have $m'(t) = -\frac{10}{2000}m(t) + 10 \cdot 10 = \frac{-1}{200}m(t) + 100$.

b.

This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-.005t} + 20,000$. Because $m(0) = 80000$, we have $C = 60,000$. Thus, $m(t) = 60,000e^{-.005t} + 20,000$.

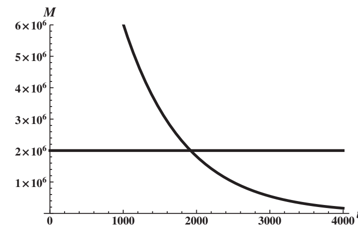


D1.5.26

a. The initial mass of the pollutant is $m_0 = 1000000 \cdot 20 = 20,000,000$. We have $m'(t) = -\frac{1200}{1000000}m(t) + 0 \cdot 1200 = \frac{-3}{2500}m(t)$.

b.

This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-.0012t}$. Because $m(0) = 20,000,000$, we have $C = 20,000,000$. Thus, $m(t) = 20,000,000e^{-.0012t}$. The pond will have a mass of 10 percent of the initial value when $.1 = e^{-.0012t}$, so $t = -\ln(.1)/.0012 \approx 1919$ hours.



D1.5.27

a. x is the predator and y is the prey.

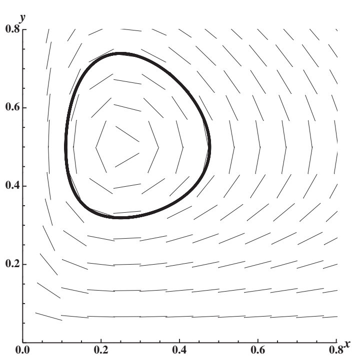
b. $-3x + 6xy = 0$ when $x = 0$ or $y = 1/2$. $y - 4xy = 0$ when $y = 0$ or $x = 1/4$.
The desired lines are $x = 1/4$ and $y = 1/2$.

c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(1/4, 1/2)$.

d. Note that $x'(t) = 3x(2y - 1)$ and $y'(t) = y(1 - 4x)$.

- For $0 < x < 1/4$ and $0 < y < 1/2$, we have $x' < 0$ and $y' > 0$.
- For $0 < x < 1/4$ and $y > 1/2$, we have $x' > 0$ and $y' > 0$.
- For $x > 1/4$ and $0 < y < 1/2$, we have $x' < 0$ and $y' < 0$.
- For $x > 1/4$ and $y > 1/2$, we have $x' > 0$ and $y' < 0$.

e. The direction of the solution is clockwise.



D1.5.28

a. x is the prey and y is the predator.

b. $2x - 4xy = 0$ when $x = 0$ or $y = 1/2$. $-y + 2xy = 0$ when $y = 0$ or $x = 1/2$.
The desired lines are $x = 1/2$ and $y = 1/2$.

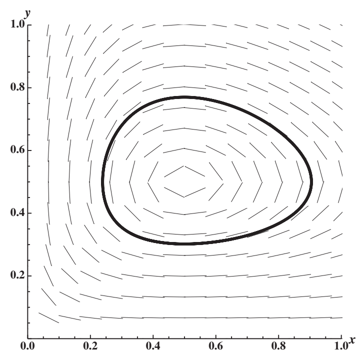
c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(1/2, 1/2)$.

d. Note that $x'(t) = 2x(1 - 2y)$ and $y'(t) = y(2x - 1)$.

- For $0 < x < 1/2$ and $0 < y < 1/2$, we have $x' > 0$ and $y' < 0$.
- For $0 < x < 1/2$ and $y > 1/2$, we have $x' < 0$ and $y' < 0$.

- For $x > 1/2$ and $0 < y < 1/2$, we have $x' > 0$ and $y' > 0$.
- For $x > 1/2$ and $y > 1/2$, we have $x' < 0$ and $y' > 0$.

e. The direction of the solution is counterclockwise.



D1.5.29

a. x is the predator and y is the prey.

b. $-3x + xy = 0$ when $x = 0$ or $y = 3$. $2y - xy = 0$ when $y = 0$ or $x = 2$.

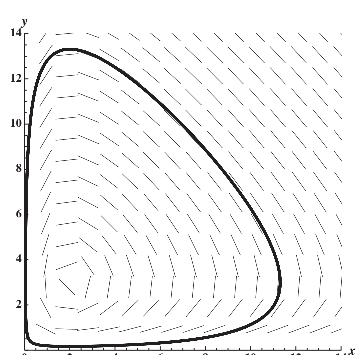
The desired lines are $x = 2$ and $y = 3$.

c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(2, 3)$.

d. Note that $x'(t) = x(y - 3)$ and $y'(t) = y(2 - x)$.

- For $0 < x < 2$ and $0 < y < 3$, we have $x' < 0$ and $y' > 0$.
- For $0 < x < 2$ and $y > 3$, we have $x' > 0$ and $y' > 0$.
- For $x > 2$ and $0 < y < 3$, we have $x' < 0$ and $y' < 0$.
- For $x > 2$ and $y > 3$, we have $x' > 0$ and $y' < 0$.

e. The direction of the solution is clockwise.



D1.5.30

a. x is the prey and y is the predator.

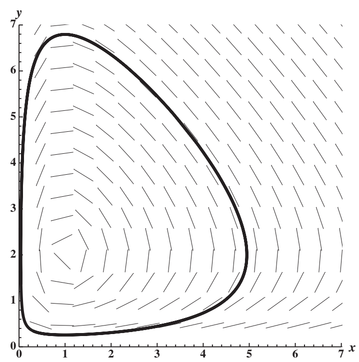
b. $2x - xy = 0$ when $x = 0$ or $y = 2$. $-y + xy = 0$ when $y = 0$ or $x = 1$.
The desired lines are $x = 1$ and $y = 2$.

c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(1, 2)$.

d. Note that $x'(t) = x(2 - y)$ and $y'(t) = y(x - 1)$.

- For $0 < x < 1$ and $0 < y < 2$, we have $x' > 0$ and $y' < 0$.
- For $0 < x < 1$ and $y > 2$, we have $x' < 0$ and $y' < 0$.
- For $x > 1$ and $0 < y < 2$, we have $x' > 0$ and $y' > 0$.
- For $x > 1$ and $y > 2$, we have $x' < 0$ and $y' > 0$.

e. The direction of the solution is counterclockwise.



D1.5.31

a. True. The growth rate function is the derivative, so where it is positive, the population is increasing.

b. True. In the limit, the solution in the tank is the same as the solution being poured in.

c. True. In the absence of predators, we assume that the prey population increases exponentially.

D1.5.32

a. $f'(P) = r\left(1 - \frac{P}{K}\right) + rP\left(\frac{-1}{K}\right) = r\left(1 - \frac{2P}{K}\right)$. This is zero when $P = \frac{K}{2}$, and an analysis of the sign of f' shows that $f' > 0$ for $P < \frac{K}{2}$ and $f' < 0$ when $P > \frac{K}{2}$. Thus there is a maximum of $f(K/2) = r(K/2)(1/2) = rK/4$ at $P = K/2$.

b. $f'(M) = -r\ln(M/K) - rM(K/M)(1/K) = -r(\ln(M/K) + 1)$. This is zero when $\ln(M/K) = -1$, or $M/K = e^{-1}$, which occurs for $M = \frac{K}{e}$. Note that $f(K/e) = -r(K/e)\ln(1/e) = rK/e$. An analysis of the sign of f' shows that $f' > 0$ for $M < \frac{K}{e}$ and $f' < 0$ when $M > \frac{K}{e}$. Thus there is a maximum at $M = K/e$.

D1.5.33 We must solve $P'(t) = rP\left(1 - \frac{P}{K}\right)$. We have $\int \frac{KP'(t)}{P(K-P)}\,dt = \int r\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{K-P(t)}\right) dt = \int r\,dt$. Thus, $\ln\left(\frac{P(t)}{K-P(t)}\right) = rt + C$. Taking the exponential of both sides and reciprocating gives $\frac{K-P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = P_0$, we have $A = \frac{K-P_0}{P_0}$. Thus $\frac{K}{P(t)} = (\frac{K-P_0}{P_0})e^{-rt} + 1$, so $P(t) = \frac{K}{\frac{K-P_0}{P_0}e^{-rt}+1} = \frac{KP_0}{(K-P_0)e^{-rt}+P_0}$.

D1.5.34 $M(0) = K\left(\frac{M_0}{K}\right)^{e^0} = K\left(\frac{M_0}{K}\right) = M_0$. Because $\lim_{t\to\infty} e^{-rt} = 0$, we have $\lim_{t\to\infty} M(t) = K\left(\frac{M_0}{K}\right)^0 = K$.

D1.5.35

a. Note that this differential equation is first-order linear with $k = \frac{-R}{V}$ and $b = C_iR$, so the solution is $m(t) = Ce^{(-R/V)t} - \frac{C_iR}{-R/V} = Ce^{(-R/V)t} + C_iV$. Because $m(0) = m_0$, we have $m_0 = C + C_iV$, so $C = m_0 - C_iV$. Thus, the solution is $m(t) = (m_0 - C_iV)e^{-Rt/V} + C_iV$.

b. $m(0) = (m_0 - C_iV) + C_iV = m_0$.

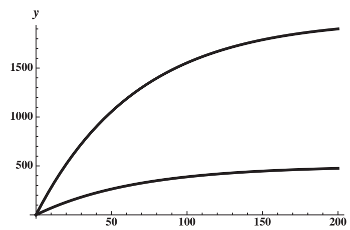
c. Note that $\lim_{t\to\infty} e^{-Rt/V} = 0$, so $\lim_{t\to\infty} m(t) = C_iV$. In the limit, the solution in the tank is the incoming solution, so the amount of material in the tank is the amount per unit volume times the volume of the tank.

d. Increasing R causes the graph to approach the asymptote more quickly.

D1.5.36

a. The initial mass of the drug in the body is 0. We have $m'(t) = -\frac{.06}{4}m(t) + (.06)(500) = -.015m(t) + 30$.

b. This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-.015t} + 2000$. Because $m(0) = 0$, we have $C = -2000$. Thus, $m(t) = -2000e^{-.015t} + 2000$.



c. Because $\lim_{t\to\infty} e^{-.015t} = 0$, $\lim_{t\to\infty} m(t) = 2000$.

d. The drug mass reaches 1800 mg when $1800 = -2000e^{-.015t} + 2000$, so when $.1 = e^{-.015t}$, so $t = -\ln(.1)/.015 \approx 153.5$ minutes.

D1.5.37

a. We can write the equation as $I'(t) = \frac{-1}{RC}I(t)$. This is a first-order linear equation with $k = \frac{-1}{RC}$ and $b = 0$. Thus, the solution is $I(t) = Ce^{kt} - \frac{b}{k} = Ae^{-t/RC}$. Because $I(0) = \frac{V}{R}$, we have $A = \frac{V}{R}$. Thus, $I(t) = \frac{V}{R}e^{-t/RC}$. The current decays exponentially with decay constant $\frac{-1}{RC}$.

b. We can write the equation as $Q'(t) = \frac{-1}{RC}Q + \frac{V}{R}$. This is a first-order linear equation with $k = \frac{-1}{RC}$ and $b = \frac{V}{R}$. Thus, the solution is $Q(t) = Ce^{kt} - \frac{b}{k} = Ae^{-t/RC} - \frac{(V/R)}{(-1/RC)} = Ae^{-t/RC} + VC$. Because $Q(0) = 0$, we have $A = -VC$. Thus, $Q(t) = (-VC)\,e^{-t/RC} + VC$. In the long run, the charge has limit VC.

D1.5.38

a. $P(t) = P_0e^{rt} = 281e^{rt}$. Because $P(10) = 281e^{10r} = 310$, we have $r = \ln(310/281)/10 \approx .0098$.

b. If $P(t) = \frac{KP_0}{(K-P_0)e^{-rt}+P_0} = \frac{281K}{(K-281)e^{-.0098t}+281}$, and if $P(50) = 439$, then $\frac{281K}{(K-281)e^{-.0098(50)}+281} = 439$. Solving for K gives $K \approx 3963$.

c. It will reach 95 percent of 3963 (which is about 3765) when $\frac{3963\cdot 281}{(3963-281)e^{-.0098t}+281} = 3765$. Solving for t gives $t \approx 562.5$.

d. If $P(t) = \frac{KP_0}{(K-P_0)e^{-rt}+P_0} = \frac{281K}{(K-281)e^{-.0098t}+281}$, and if $P(50) = 450$, then $\frac{281K}{(K-281)e^{-.0098(50)}+281} = 450$. Solving for K gives $K \approx 9211$.

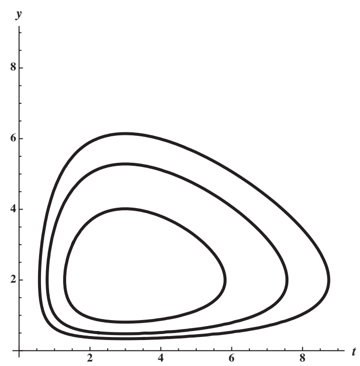
e. If $P(t) = \frac{KP_0}{(K-P_0)e^{-rt}+P_0} = \frac{281K}{(K-281)e^{-.0098t}+281}$, and if $P(50) = 430$, then $\frac{281K}{(K-281)e^{-.0098(50)}+281} = 430$. Solving for K gives $K \approx 2664$.

f. Small differences in the 40-year projection result in huge differences in the estimated carrying capacity.

D1.5.39

a. We have $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{cy-dxy}{-ax+bxy} = \frac{y(c-dx)}{x(by-a)}$. Thus we can write $\frac{by-a}{y}\frac{dy}{dx} = \frac{c-dx}{x}$, or $\left(b - \frac{a}{y}\right)\frac{dy}{dx} = \frac{c}{x} - d$. Thus, the equation is separable.

b. We have $\int \left(b - \frac{a}{y}\right)\frac{dy}{dx}\,dx = \int \left(\frac{c}{x} - d\right)\,dx$, so $by - a\ln y = c\ln x - dx + K$, so $dx + by = c\ln x + a\ln y + K$. Taking the exponential function of both sides gives $e^{dx+by} = Cx^c y^a$ for an arbitrary constant C.

c. 

D1.6 Chapter Review

1

a. False. It is a first-order, linear differential equation, but it isn't separable.

b. False. It is a first-order, separable differential equation, but it isn't linear.

c. True. Note that $y' = 1 - t^{-2}$, so $ty' = t - t^{-1}$. Thus, $ty' + y = t - t^{-1} + (t + t^{-1}) = 2t$. Also, $y(1) = 2$.

d. True.

e. False. In general, Euler's method gives approximate solutions.

2 $y'(t) = -3y$, so $\int \frac{y'(t)}{y}\,dt = \int(-3)\,dt$. Thus, $\ln|y| = -3t + K$, and exponentiating both sides gives $y = Ce^{-3t}$.

3 $y'(t) = -2y + 6$, so $\int \frac{y'(t)}{-2y+6}\,dt = \int 1\,dt$, and therefore $\frac{-1}{2}\ln|-2y+6| = t + K$. It follows that $|-2y+6| = Ae^{-2t}$. We can write $-2y + 6 = \pm Ae^{-2t}$, so $y(t) = Ce^{-2t} + 3$.

4 $p'(x) = 4p + 8 = 4(p+2)$, so $\int \frac{p'(x)}{p+2}\,dx = \int 4\,dx$, and thus $\ln|p+2| = 4x + K$. Thus $|p+2| = Ae^{4x}$, so $p + 2 = \pm Ae^{4x}$, and thus $p(x) = Ce^{4x} - 2$.

5 $y'(t) = 2ty$, so $\int \frac{y'(t)}{y}\,dt = \int 2t\,dt$, and therefore, $\ln|y| = t^2 + K$. It follows that $y = Ce^{t^2}$.

6 $y'(t) = \frac{\sqrt{y}}{\sqrt{t}}$, so $\int y^{-1/2}y'(t)\,dt = \int t^{-1/2}\,dt$. Integrating gives $2\sqrt{y} = 2\sqrt{t} + K$, so $\sqrt{y} = \sqrt{t} + C$, so $y(t) = (\sqrt{t} + C)^2$.

7 $\frac{y'(t)}{y} = \frac{1}{t^2+1}$. Integrating both sides with respect to t gives $\ln|y| = \tan^{-1}(t) + K$, so $y = Ce^{\tan^{-1}(t)}$.

8 $2yy'(x) = \sin x$, so $y^2 = -\cos x + C$, and thus $y = \pm\sqrt{C - \cos x}$.

9 $\int \frac{y'(t)}{y^2+1}\,dt = \int (2t+1)\,dt$, so $\tan^{-1}(y) = t^2 + t + C$, and thus $y = \tan(t^2 + t + C)$.

10 $\frac{z'(t)}{z} = \frac{t}{t^2+1}$. Integrating both sides with respect to t gives $\ln|z| = \frac{1}{2}\ln|t^2+1| + K$. Thus $z = C\sqrt{t^2+1}$.

11 $\int y'(t)\,dt = \int (2t + \cos t)\,dt$, so $y(t) = t^2 + \sin t + C$. Because $y(0) = 1$, we have $1 = 0 + 0 + C$, so $C = 1$. Thus, $y(t) = t^2 + \sin t + 1$.

12 $y'(t) = -3(y-3)$, so $\int \frac{y'(t)}{y-3}\,dt = \int (-3)\,dt$, so $\ln|y-3| = -3t + K$, and thus $y - 3 = Ce^{-3t}$. Because $y(0) = 4$, we have $C = 1$. Thus $y = e^{-3t} + 3$.

13 $\int \frac{Q'(t)}{Q-8}\,dt = \int 1\,dt$, so $\ln|Q-8| = t + K$. Thus, $Q - 8 = Ce^t$. Because $Q(1) = 0$, we have $-8 = Ce$, so $C = \frac{-8}{e}$. Thus, $Q = -8e^{t-1} + 8$.

14 $yy'(x) = x$, so $\int yy'(x)\,dx = \int x\,dx$, so $y^2/2 = x^2/2 + C$. Because $y(2) = 4$, we have $8 = 2 + C$, so $C = 6$. Thus, $y^2 = x^2 + 12$, and $y = \sqrt{x^2+12}$.

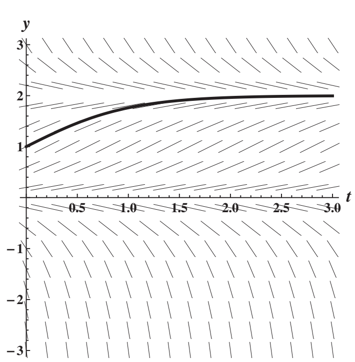
15 $u^{-1/3}u'(t) = t^{-1/3}$. Thus $\int u^{-1/3}u'(t)\,dt = \int t^{-1/3}\,dt$. We have $\frac{3}{2}u^{2/3} = \frac{3}{2}t^{2/3} + C$. Because $u(1) = 8$, we have $6 = \frac{3}{2} + C$, so $C = \frac{9}{2}$. Thus, $u^{2/3} = t^{2/3} + 3$, and $u = (t^{2/3}+3)^{3/2}$.

16 $\int (\sin y)y'(x)\,dx = \int 4x\,dx$, so $-\cos y = 2x^2 + C$. Because $y(0) = \pi/2$, we have $0 = 0 + C$, so $C = 0$. Thus $\cos y = -2x^2$, and $y = \cos^{-1}(-2x^2)$.

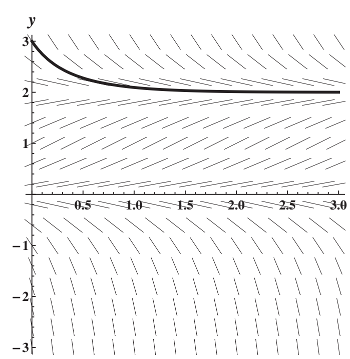
17 $\int (2s)s'(t)\,dt = \int \frac{dt}{t+2}$. Thus, $s^2 = \ln(t+2) + C$. Because $s(-1) = 4$, we have $16 = 0 + C$, so $s^2 = \ln(t+2) + 16$, and $s = \sqrt{\ln(t+2) + 16}$.

18 $\int \sec^2\theta\,\theta'(x)\,dx = \int 4x\,dx$. Thus, $\tan\theta = 2x^2 + C$. Because $\theta(0) = \pi/4$, we have $1 = 0 + C$, so $C = 1$. It follows that $\theta = \tan^{-1}(2x^2 + 1)$.

19



a.



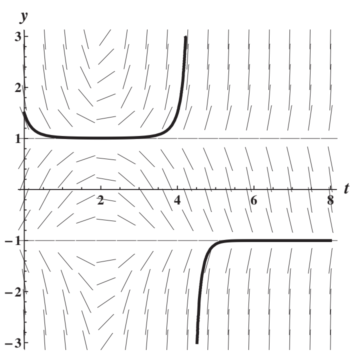
b.

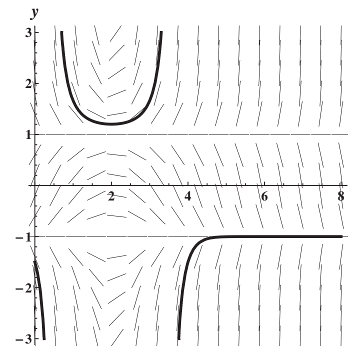
c. The solutions are increasing when $0 < A < 2$.

d. The solutions are decreasing when $A < 0$ or when $A > 2$.

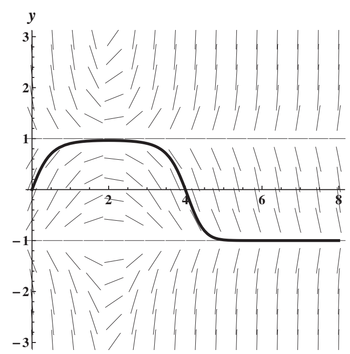
e. The equilibrium solutions are $y = 0$ and $y = 2$.

20

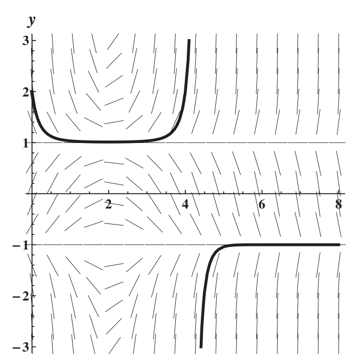
a. 

b. 

c. It appears that $\lim_{t\to\infty} y(t) = -1$.



d. It appears that $\lim_{t\to\infty} y(t) = -1$.



e. The equilibrium solutions are $y = 1$ and $y = -1$.

21

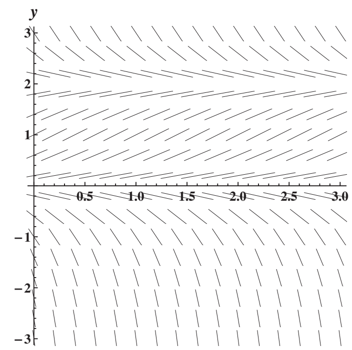
a. $u_0 = 1$. $u_1 = 1 + f(0,1)(.1) = 1 + 1/20 = 1.05$. Also, $u_2 = 1.05 + f(.1, 1.05)(.1) = 1.05 + .047619 = 1.09762$. Thus, $y(.1) \approx 1.05$ and $y(.2) \approx 1.09762$.

b. $u_0 = 1$. $u_1 = 1 + f(0,1)(.05) = 1 + .025 = 1.025$. Also, $u_2 = 1.025 + f(.05, 1.025)(.05) = 1.04939$. $u_3 = 1.04939 + f(.1, 1.04939)(.05) = 1.07321$. $u_4 = 1.07321 + f(.15, 1.07321)(.05) = 1.0961$. Thus, $y(.1) \approx 1.04939$ and $y(.2) \approx 1.09651$.

c. For part a, we have $\frac{|1.09762-\sqrt{1.2}|}{\sqrt{1.2}} = 0.00198539$. For part b, we have $\frac{|109651-\sqrt{1.2}|}{\sqrt{1.2}} = 0.000972103$. Part b is more accurate.

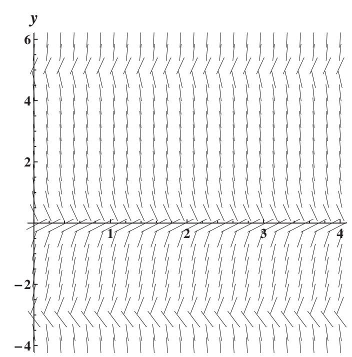
22

The equilibrium solutions are $y = 0$ (unstable) and $y = 2$ (stable).



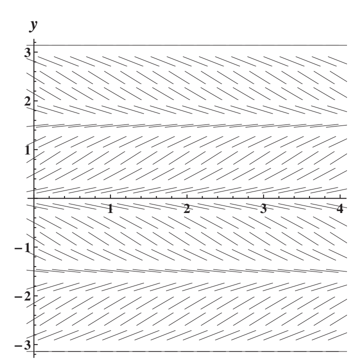
23

The equilibrium solutions are $y = -3$ (unstable) and $y = 0$ (stable), and $y = 5$ (unstable).



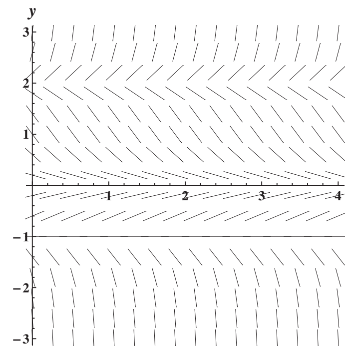
24

The equilibrium solutions are $y = -\pi/2$ (stable) and $y = 0$ (unstable), and $y = \pi/2$ (stable).



25

The equilibrium solutions are $y = -1$ (unstable) and $y = 0$ (stable), and $y = 2$ (unstable).



26

a. The equilibrium solutions occur where $P'(t) = 0$, so they are $P = 0$ and $P = 1200$.

b. We must solve $P'(t) = .2P\left(1 - \frac{P}{1200}\right)$. We have $\int \frac{1200P'(t)}{P(1200-P)}\,dt = \int .2\,dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{1200-P(t)}\right)dt = \int .2\,dt$. Thus, $\ln\left(\frac{P(t)}{1200-P(t)}\right) = .2t + C$. Taking the exponential of both sides and reciprocating gives $\frac{1200-P(t)}{P(t)} = Ae^{-.2t}$. Because $P(0) = 50$, we have $A = 23$. Thus $\frac{1200}{P(t)} = 23e^{-.2t} + 1$, so $P(t) = \frac{1200}{23e^{-.2t}+1} = \frac{60000}{1150e^{-.2t}+50}$.

c. The carrying capacity is $\lim_{t\to\infty} \frac{1200}{23e^{-.2t}+1} = 1200$.

d. $P''(t) = .2(1 - (P/1200)) + .2P(-1/1200) = .2 - .2(P/600)$. This is zero when $P = 600$. Also, note that $P'''(600) < 0$, so there is a maximum for P' at 600 by the Second Derivative Test.

27

a. The logistic equation has solution $P(t) = \frac{KP_0}{(K-P_0)e^{-rt}+P_0} = \frac{32000}{1580e^{-rt}+20}$. We are seeking r so that $80 = \frac{32000}{1580e^{-20r}+20}$. Solving for r gives $r = \frac{\ln(79/19)}{20} \approx .0713$.

b. The solution is $P(t) = \frac{32000}{1580e^{-.0713t}+20} = \frac{1600}{79e^{-.0713t}+1}$.

c. We are seeking t so that $800 = \frac{1600}{79e^{-.0713t}+1}$. Solving for t gives $t \approx 61$ hours.

28

a. Assuming $P(t) = 435e^{kt}$ and $P(5) = 435e^{5k} = 487$, we have $k = \ln(487/435)/5 \approx .0226$.

b. The logistic equation has solution $P(t) = \frac{KP_0}{(K-P_0)e^{-rt}+P_0} = \frac{K\cdot 435}{(K-435)e^{-.0226t}+435}$. We are seeking K so that $1570 = \frac{K\cdot 435}{(K-435)e^{-.0226\cdot 90}+435}$. Solving for K gives $K \approx 2585$.

c. $P(t) = \frac{2585\cdot 435}{2150e^{-.0226t}+435}$.

d. Solving $2000 = \frac{2585\cdot 435}{2150e^{-.0226t}+435}$ for t gives $t \approx 125$, which would be in about 2085.

e. It is very hard to predict the actual carrying capacity. Perhaps the estimate for the population in 2050 isn't accurate. Also, small changes in these kinds of estimates can lead to large changes in the actual population sizes.

29

a. The initial mass of the sugar is $m_0 = 0$. We have $m'(t) = -\frac{.5}{100}m(t) + 100\cdot .5 = \frac{-1}{200}m(t) + 10$. This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-.005t} + 2000$. Because $m(0) = 0$, we have $C = -2000$. Thus, $m(t) = -2000e^{-.005t} + 2000$.

b. The steady-state mass is $\lim_{t\to\infty} m(t) = 2000$.

c. The mass reaches .95(2000) when $.05 = e^{-.005t}$, or $t = \ln(.05)/(-.005) \approx 599$ minutes.

30

a. $T(t) = (80-25)e^{-kt} + 25$. Because $T(5) = 60$, we have $60 = 55e^{-5k} + 25$, so $35/55 = e^{-5k}$, and thus $k = -\ln(35/55)/5 \approx .0904$.

b. $T(t) = 55e^{-.0904t} + 25$.

c. The coffee reaches 50 degrees when $50 = 55e^{-.0904t} + 25$, or when $e^{-.0904t} = 25/55$. Thus $t = -\ln(5/11)/.0904 \approx 8.7$ minutes.

31

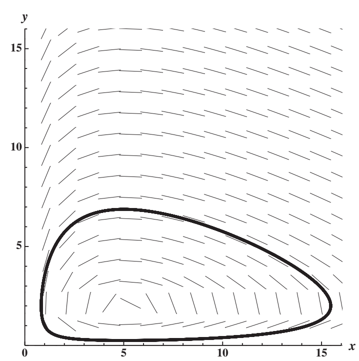
a. x represents the predator and y represents the prey.

b. $x'(t) = 2x(-2+y)$ is zero when $x = 0$ and when $y = 2$. $y'(t) = y(5-x)$ is zero when $y = 0$ and when $x = 5$.

c. The equilibrium points occur when both equations are zero simultaneously, which occurs at $(0,0)$ and $(5,2)$.

d.
- For $0 < x < 5$ and $0 < y < 2$, we have $x' < 0$ and $y' > 0$.
- For $0 < x < 5$ and $y > 2$, we have $x' > 0$ and $y' > 0$.
- For $x > 5$ and $0 < y < 2$, we have $x' < 0$ and $y' < 0$.
- For $x > 5$ and $y > 2$, we have $x' > 0$ and $y' < 0$.

e. The solution evolves in the clockwise direction.



32

a. $t^2y'(t) + (2t)y(t) = \frac{d}{dt}(t^2y(t))$.

b. $\int(t^2y'(t) + (2t)y(t))\,dt = \int e^{-t}\,dt$, so $t^2y(t) = -e^{-t} + C$. Thus, $y(t) = \frac{-e^{-t}+C}{t^2}$.

c. If $y(1) = 0$, then $0 = C - 1/e$, so $C = 1/e$. Thus, $y(t) = \frac{-e^{-t}+1/e}{t^2}$.

33

a. If $y(t) = t^p$ is a solution, then $p(p-1)t^p + 2pt^p - 12t^p = 0$, so $p^2 - p + 2p - 12 = 0$, so $p^2 + p - 12 = (p-3)(p+4) = 0$. Thus, $p_1 = 3$ and $p_2 = -4$.

b. If $y(t) = C_1t^3 + C_2t^{-4}$, and if $y(1) = 0$, then $C_1 + C_2 = 0$. Note that $y'(t) = 3C_1t^2 - 4C_2t^{-5}$, so if $y'(1) = 7$, we have $7 = 3c_1 - 4C_2$. Solving the system of linear equations in C_1 and C_2 gives $C_1 = 1$ and $C_2 = -1$. Thus the solution we seek is $y(t) = t^3 - t^{-4}$.